

# HOMOGENEITY GROUPS OF ENDS OF OPEN 3-MANIFOLDS

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ABSTRACT. For every finitely generated abelian group  $G$ , we construct an irreducible open 3-manifold  $M_G$  whose end set is homeomorphic to a Cantor set and with end homogeneity group of  $M_G$  isomorphic to  $G$ . The end homogeneity group is the group of self-homeomorphisms of the end set that extend to homeomorphisms of the 3-manifold. The techniques involve computing the embedding homogeneity groups of carefully constructed Antoine type Cantor sets made up of rigid pieces. In addition, a generalization of an Antoine Cantor set using infinite chains is needed to construct an example with integer homogeneity group. Results about local genus of points in Cantor sets and about geometric index are also used.

## 1. INTRODUCTION

Each Cantor set  $C$  in  $S^3$  has complement an open 3-manifold  $M^3$  with end set  $C$ . Properties of the embedding of the Cantor set give rise to properties of the corresponding complementary 3-manifold  $M^3$ . See [SS12], [GR12], and [GRW12] for examples of this.

We investigate possible group actions on the end set  $C$  of the open 3-manifold  $M^3$  in the following sense: The *homogeneity group of the end set* is the group of homeomorphisms of the end set  $C$  that extend to homeomorphisms of the open 3-manifold  $M^3$ . Referring specifically

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to the embedding of the Cantor set, this group can also be called the *embedding homogeneity group of the Cantor set*. See [Dij10] and [vM11] for a discussion and overview of some other types of homogeneity.

The standardly embedded Cantor set is at one extreme here. The embedding homogeneity group is the full group of self-homeomorphisms of the Cantor set, an extremely rich group (there is such a homeomorphism taking any countable dense set to any other). Cantor sets with this full embedding homogeneity group are called *strongly homogeneously embedded*. See Daverman [Dav79] for an example of a non-standard Cantor set with this property.

At the other extreme are *rigidly embedded* Cantor sets, i.e. those Cantor sets for which only the identity homeomorphism extends. Shilepisky [Shi74] constructed Antoine type [Ant20] rigid Cantor sets. Their rigidity is a consequence of Sher's result [She68] that if two Antoine Cantor sets are equivalently embedded, then the stages of their defining sequences must match up exactly. In the last decade, new examples [GRŽ06, GRWŽ11] of non-standard Cantor sets were constructed that were both rigidly embedded and had simply connected complement. See [Wri86] for additional examples of rigidity.

These examples naturally lead to the question of which types of groups can arise as end homogeneity groups between the two extremes mentioned above. In this paper we show that for each finitely generated abelian group  $G$ , there is an irreducible open three-manifold with the end set homeomorphic to a Cantor set and with the end homogeneity group isomorphic to  $G$ . (See Corollary 6.3.)

The Cantor sets produced are *unsplittable*, in the sense that for each such  $C$ , no 2-sphere in the complement of  $C$  separates points of  $C$ . We produce these examples by constructing, for each natural number  $m$  greater than one, 3-manifolds with end homogeneity groups  $\mathbb{Z}_m$ , and by separately constructing 3-manifolds with end homogeneity group  $\mathbb{Z}$ .

We then link the Cantor sets needed for a given abelian group in an unsplitable manner.

In Section 2, we give definitions and the basic results needed for the rest of the paper. In the following section, Section 3, we review the needed results about Antoine Cantor sets. In Section 4 we produce Cantor sets with embedding homogeneity group  $\mathbb{Z}_m$ . In Section 5 we produce Cantor sets with embedding homogeneity group  $\mathbb{Z}$ . Section 6 ties together the previous results and lists and proves the main theorems. Section 7 lists some remaining questions.

## 2. PRELIMINARIES

**Background information.** Refer to [GRŽ05, GRŽ06, GRW12] for a discussion of Cantor sets in general and of rigid Cantor sets, and to [Žel05] for results about local genus of points in Cantor sets and defining sequences for Cantor sets. The bibliographies in these papers contain additional references to results about Cantor sets. Two Cantor sets  $X$  and  $Y$  in  $S^3$  are said to be *equivalent* if there is a self-homeomorphism of  $S^3$  taking  $X$  to  $Y$ . If there is no such homeomorphism, the Cantor sets are said to be *inequivalent*, or *inequivalently embedded*. A Cantor set  $C$  is *rigidly embedded* in  $S^3$  if the only self-homeomorphism of  $C$  that extends to a homeomorphism of  $S^3$  is the identity.

**Geometric index.** We list the results we need on geometric index. See Schubert [Sch53] and [GRWŽ11] for more details.

If  $K$  is a link in the interior of a solid torus  $T$ , the *geometric index* of  $K$  in  $T$  is denoted by  $N(K, T)$ . This *geometric index* is the minimum of  $|K \cap D|$  over all meridional disks  $D$  of  $T$  intersecting  $K$  transversely. If  $T$  is a solid torus and  $M$  is a finite union of disjoint solid tori so that  $M \subset \text{Int}(T)$ , then the geometric index  $N(M, T)$  of  $M$  in  $T$  is  $N(K, T)$  where  $K$  is a core of  $M$ .

**Theorem 2.1.** ([Sch53], [GRWŽ11, Theorem 3.1]) *Let  $T_0$  and  $T_1$  be unknotted solid tori in  $S^3$  with  $T_0 \subset \text{Int}(T_1)$  and  $N(T_0, T_1) = 1$ . Then  $\partial T_0$  and  $\partial T_1$  are parallel; i.e., the manifold  $T_1 - \text{Int}(T_0)$  is homeomorphic to  $\partial T_0 \times I$  where  $I$  is the closed unit interval  $[0, 1]$ .*

**Theorem 2.2.** ([Sch53], [GRWŽ11, Theorem 3.2]) *Let  $T_0$  be a finite union of disjoint solid tori. Let  $T_1$  and  $T_2$  be solid tori so that  $T_0 \subset \text{Int}(T_1)$  and  $T_1 \subset \text{Int}(T_2)$ . Then  $N(T_0, T_2) = N(T_0, T_1) \cdot N(T_1, T_2)$ .*

There is one additional result we will need.

**Theorem 2.3.** ([Sch53], [GRWŽ11, Theorem 3.3]) *Let  $T$  be a solid torus in  $S^3$  and let  $T_1, \dots, T_n$  be unknotted pairwise disjoint solid tori in  $T$ , each of geometric index 0 in  $T$ . Then the geometric index of  $\bigcup_{i=1}^n T_i$  in  $T$  is even.*

**Defining sequences and local genus.** We review the definition and some basic facts from [Žel05] about the local genus of points in a Cantor set. See [Žel05] for a discussion of defining sequences.

Let  $\mathcal{D}(X)$  be the set of all defining sequences for a Cantor set  $X \subset S^3$ . Let  $(M_i) \in \mathcal{D}(X)$  be a specific defining sequence for a  $X$ . For  $A \subset X$ , denote by  $M_i^A$  the union of those components of  $M_i$  which intersect  $A$ . The genus of  $M_i^A$ ,  $g(M_i^A)$ , is the maximum genus of a component of  $M_i^A$ . Define

$$g_A(X; (M_i)) = \sup\{g(M_i^A); i \geq 0\} \quad \text{and}$$

$$g_A(X) = \inf\{g_A(X; (M_i)); (M_i) \in \mathcal{D}(X)\}.$$

The number  $g_A(X)$  is called *the genus of the Cantor set  $X$  with respect to the subset  $A$* . For  $A = \{x\}$  we call the number  $g_{\{x\}}(X)$  *the local genus of the Cantor set  $X$  at the point  $x$*  and denote it by  $g_x(X)$ .

Let  $x$  be an arbitrary point of a Cantor set  $X$  and  $h: S^3 \rightarrow S^3$  a homeomorphism. Then the local genus  $g_x(X)$  is the same as the local genus  $g_{h(x)}(h(X))$ . Also note that if  $x \in C \subset C'$ , then the local genus

of  $x$  in  $C$  is less than or equal to the local genus of  $x$  in  $C'$ . See [Žel05, Theorem 2.4].

The following result from [Žel05] is needed to show that certain points in our examples have local genus 2.

**Theorem 2.4.** [Žel05] *Let  $X, Y \subset S^3$  be Cantor sets and  $p \in X \cap Y$ . Suppose there exists a 3-ball  $B$  and a 2-disc  $D \subset B$  such that*

- (1)  $p \in \text{Int}(B)$ ,  $\partial D = D \cap \partial B$ ,  $D \cap (X \cup Y) = \{p\}$ ; and
- (2)  $X \cap B \subset B_X \cup \{p\}$  and  $Y \cap B \subset B_Y \cup \{p\}$  where  $B_X$  and  $B_Y$  are the components of  $B - D$ .

*Then  $g_p(X \cup Y) = g_p(X) + g_p(Y)$ .*

**Discussion and examples of ends and homogeneity groups.** For background on Freudenthal compactifications and theory of ends, see [Dic68], [Fre42], and [Sie65]. For an alternate proof using defining sequences of the result that every homeomorphism of the open 3-manifold extends to a homeomorphism of its Freudenthal compactification, see [GR12].

At the end of the next section, we will discuss elements of the homogeneity group of a standard self-similar Antoine Cantor set. Note that removing  $n$  points from  $S^3$  yields a reducible open 3-manifold with end homogeneity group the symmetric group on  $n$  elements. It is not immediately obvious how to produce examples that are irreducible, have a rich end structure (for example a Cantor set), and at the same time have specified abelian end homogeneity groups.

### 3. ANTOINE CANTOR SET PROPERTIES

An Antoine Cantor set is described by a defining sequence  $(M_i)$  as follows. Let  $M_0$  be an unknotted solid torus in  $S^3$ . Let  $M_1$  be a chain of at least four linked, pairwise disjoint, unknotted solid tori in  $M_0$  as in Figure 1. Inductively,  $M_i$  consists of pairwise disjoint solid tori in  $S^3$  and  $M_{i+1}$  is obtained from  $M_i$  by placing a chain of at least four

linked, pairwise disjoint, unknotted solid tori in each component of  $M_i$ . If the diameter of the components goes to 0, the Antoine Cantor set is  $C = \bigcap_{i=0}^{\infty} M_i$ .

We refer to Sher's paper [She68] for basic results and description of Antoine Cantor sets. The key result we shall need is the following:

**Theorem 3.1.** [She68, Theorems 1 and 2] *Suppose  $C$  and  $D$  are Antoine Cantor sets in  $S^3$  with defining sequences  $(M_i)$  and  $(N_i)$  respectively. The Cantor sets are equivalently embedded if and only if there is a self-homeomorphism  $h$  of  $S^3$  with  $h(M_i) = N_i$  for each  $i$ .*

In particular, the number and adjacency of links in the chains must match up at each stage. Because we need a modification of this result for infinite chains in Section 5, we provide below an outline of an alternative proof of the forward implication.

*Proof.* (Forward implication of Theorem 3.1.) It suffices to show if  $C$  has two Antoine defining sequences  $(M_i)$  and  $(N_i)$ , then there is a homeomorphism  $h$  as in the theorem.

**Step 1:** There is a general position homeomorphism  $h_1$ , fixed on  $C$ , so that  $h_1(\partial(M_1) \cup \partial(M_2))$  is in general position with  $\partial(N_1) \cup \partial(N_2)$ . The curves of intersection of  $h_1(\partial(M_1) \cup \partial(M_2)) \cap (\partial(N_1) \cup \partial(N_2))$  can be eliminated by a homeomorphism  $h_2$  also fixed on  $C$  by a standard argument and the facts that any nontrivial curve on  $\partial(M_i)$  does not bound a disc in the complement of  $C$ , and that no 2-sphere separates the points of  $C$ . For details on the type of argument in this step, see [She68] or [GRWŽ11].

**Step 2:** Let  $T$  be a component of  $h_2 \circ h_1(M_1)$  and assume  $T$  intersects a component  $S$  of  $N_1$ . Either  $T \subset \text{Int}(S)$  or  $S \subset \text{Int}(T)$ . First assume  $T \subset \text{Int}(S)$ . If the geometric index of  $T$  in  $S$  is 0, then since the other components of  $h_2 \circ h_1(M_1)$  are linked to  $T$  by a finite chain,

all components of  $h_2 \circ h_1(M_1)$  are in the interior of  $S$ . This is a contradiction since there are points of  $C$  not in  $S$ . So the geometric index of  $T$  in  $S$  is greater than or equal to 1.

Note that  $T$  cannot be contained in any component of  $N_2$  that is in  $S$  since these have geometric index 0 in  $S$ . So  $T$  contains all the components of  $N_2$  that are in  $S$ . Each of these components has geometric index 0 in  $T$ , so the union of these components has an even geometric index in  $T$  by Theorem 2.3. This geometric index must then be 2 and the geometric index of  $T$  in  $S$  must be 1. Now there is a homeomorphism  $h_3$  fixed on  $C$  and the complement of  $S$  that takes  $T$  to  $S$ .

If instead  $S \subset \text{Int}(T)$ , a similar argument shows there is a homeomorphism  $h_3$  fixed on  $C$  and the complement of  $T$  taking  $S$  to  $T$ . The net result is that it is possible to construct a homeomorphism  $h'_3$  taking the components of  $h_2 \circ h_1(M_1)$  to the components of  $S$ . One now proceeds inductively, matching up further stages in the constructions, obtaining the desired homeomorphism  $h$  as a limit.  $\square$

**Remark 3.2.** A standard argument shows that an Antoine Cantor set cannot be separated by a 2-sphere. This is also true if the construction starts with a finite open chain of linked tori as in Figure 4.

**Remark 3.3.** Also note that the homeomorphism of Theorem 3.1 can be realized as the final stage of an isotopy since each of the homeomorphisms in the argument can be realized by an isotopy.

**Homogeneity groups of Antoine Cantor sets.** Let  $C$  be obtained by a standard Antoine construction where the same number of tori are used in tori of previous stages in each stage of the construction. For example, the Antoine pattern in Figure 1 with 24 smaller tori, each geometrically similar to the outer torus, can be repeated in each component at each stage of the construction.

We now consider some elements of the embedding homogeneity group of  $C$ . There is an obvious  $\mathbb{Z}_{24}$  group action on the resulting Cantor set obtained by rotating and twisting the large torus. There is also  $\mathbb{Z}_{24} \oplus \mathbb{Z}_{24}$  action on  $C$  by considering the first two stages, where we require each torus in the second stage to rotate the same amount. If we allow the tori in the second stage to rotate different amounts, we get an even larger group action by  $\mathbb{Z}_{24}$  wreath product with  $\mathbb{Z}_{24}$ . Considering more stages results in even more complicated group actions.

In addition to these group actions by rotating and twisting, there are also orientation reversing  $\mathbb{Z}_2$  actions that arise from reflecting through a horizontal plane (containing the core of the large torus) or through a vertical plane (containing meridians of the large torus).

From this we see that even for a simple self-similar Antoine Cantor set, the embedding homogeneity group is more complex than just the group of obvious rotations from the linking structure. In the next section we shall carefully combine certain Antoine constructions to produce a more rigid example with nontrivial end homogeneity group in such a way that these kinds of orientation reversing homeomorphisms are not possible, and that also restricts the possible rotations.

#### 4. A CANTOR SET WITH EMBEDDING HOMOGENEITY GROUP $\mathbb{Z}_m$

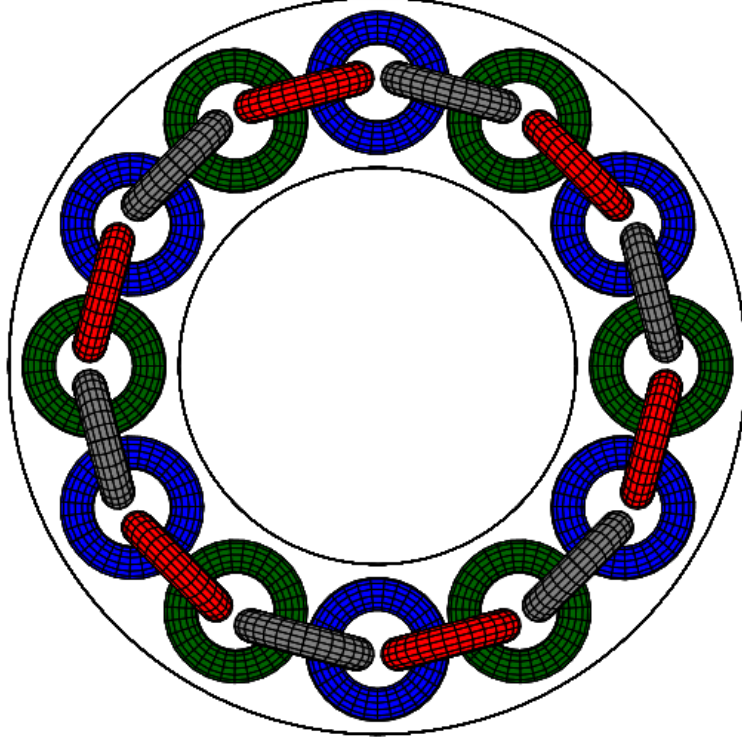
Fix an integer  $m > 1$ . We describe how to construct a Cantor set in  $S^3$  with embedding homogeneity group  $\mathbb{Z}_m$ .

**Construction 4.1.** As in the previous section, let  $S_0$  be an unknotted solid torus in  $S^3$ . Let  $\{S_{(1,i)} \mid 1 \leq i \leq 4m\}$ , be an Antoine chain of  $4m$  pairwise disjoint linked solid tori in the interior of  $S_0$  and let

$$S_1 = \bigcup_{i=1}^{4m} S_{(1,i)}.$$

See Figure 1 for the case when  $m = 6$ . Let  $C_j, 1 \leq j \leq 4$ , be a rigid Antoine Cantor set with first stage  $S_{(1,j)}$ . Choose these four



FIGURE 1. Antoine Chain With  $\mathbb{Z}_6$  Group Action

rigid Antoine Cantor sets so that they are inequivalently embedded in  $S^3$ . Let  $h$  be a homeomorphism of  $S^3$ , fixed on the complement of the interior of  $S_0$ , that takes  $S_{(1,j)}$  to  $S_{(1,j+4 \bmod 4m)}$  for  $1 \leq j \leq 4m$ . Require that  $h^m$  is the identity on each  $S_{(1,i)}$ .

For  $4k < i \leq 4k + 4$ , let  $C_i$  be the rigid Cantor set in  $S_{(1,i)}$  given by  $h^k(C_{i-4k})$ . Note that this produces  $m$  copies of each of the rigid Cantor sets  $C_1, C_2, C_3$ , and  $C_4$ . Again, see Figure 1 where the shading indicates the four classes of rigid Cantor sets. The Cantor set we are looking for is

$$C = \bigcup_{i=1}^{4m} C_i.$$

**Theorem 4.2.** *The Cantor set  $C$  from the previous construction has embedding homogeneity group  $\mathbb{Z}_m$  and is unsplittable.*

*Proof.* Let  $\ell : S^3 \rightarrow S^3$  be a homeomorphism taking  $C$  to  $C$ . We show that  $\ell|_C = h^k|_C$  for some  $k$ ,  $1 \leq k \leq m$ . By Sher [She68], we may assume that  $\ell$  takes each  $S_{(1,i)}$  to some  $S_{(1,j)}$ , and so  $\ell(C_i) = C_j$ . Because of the distinct rigid Cantor sets involved, this is only possible if  $j - i \equiv 0 \pmod{4}$ .

So assume that  $\ell(S_{(1,1)}) = S_{(1,4k+1)}$ . Then  $\ell(S_{(1,2)})$  must be one of the two tori linked with  $S_{(1,4k+1)}$ , hence either  $S_{(1,4k)}$  or  $S_{(1,4k+2)}$ . Since  $(4k-2) \not\equiv 0 \pmod{4}$ ,  $\ell(S_{(1,2)})$  must be  $S_{(1,4k+2)}$ . Continuing inductively, one sees that  $\ell(S_{(1,i)}) = S_{(1,4k+i \pmod{4m})}$ . Thus  $\ell(C_i) = C_{4k+i \pmod{4m}}$ . But  $h^k(C_i)$  is also  $C_{4k+i \pmod{4m}}$ . Since these are rigid Cantor sets,  $\ell|_{C_i} = h^k|_{C_i}$  for each  $i$ .

So the embedding homogeneity group of  $C$  is  $\{h^k | 1 \leq k \leq p\} \simeq \mathbb{Z}_m$ . By Remark 3.2,  $C$  is unsplittable. The assertion follows.  $\square$

## 5. A CANTOR SET WITH EMBEDDING HOMOGENEITY GROUP $\mathbb{Z}$

We now construct a Cantor set in  $S^3$  with embedding homogeneity group  $\mathbb{Z}$ . This requires careful analysis of an infinite chain analogue of the Antoine construction.

**Construction 5.1.** Let  $S_0$  be a pinched solid torus in  $S^3$ , i.e., the quotient of a solid torus with a meridional disk collapsed to a single point  $w$ . Let  $T_i, i \in \mathbb{Z}$  be a countable collection of unknotted pairwise disjoint solid tori in  $S_0$  so that each  $T_i$  is of simple linking type with both  $T_{i-1}$  and  $T_{i+1}$ , and is not linked with  $T_j, j \neq i-1$  or  $i+1$ . Place the tori  $T_i$  so that the  $T_i, i > 0$  and the  $T_i, i < 0$  have  $w$  as a limit point as in Figure 2.

Let  $C_j, 1 \leq j \leq 3$ , be a rigid Antoine Cantor set with first stage  $T_j$ . Choose these three rigid Antoine Cantor sets so that they are inequivalently embedded in  $S^3$ . Let  $\alpha$  be a homeomorphism of  $S^3$ , fixed on the complement of the interior of  $S_0$ , that takes  $T_j$  to  $T_{j+3}$  for  $j \in \mathbb{Z}$ .

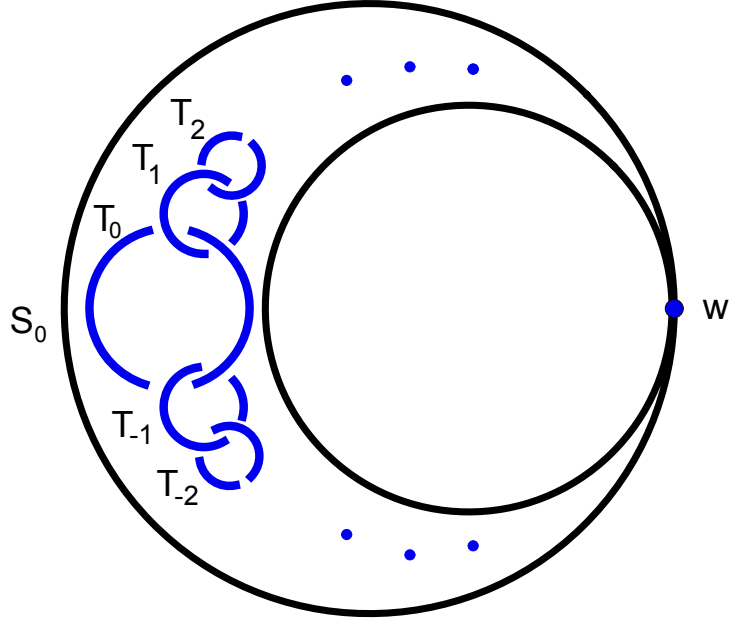


FIGURE 2. Infinite Antoine Chain

For  $3k < i \leq 3k + 3$ , let  $C_i$  be the rigid Cantor set in  $T_i$  given by  $\alpha^k(C_{i-3k})$ . Note that this produces a countable number of copies of each of the rigid Cantor sets  $C_1$ ,  $C_2$ , and  $C_3$ . Again, see Figure 2. The Cantor set we are looking for is

$$C = \overline{\bigcup_{i \in \mathbb{Z}} C_i} = \bigcup_{i \in \mathbb{Z}} C_i \cup \{w\}.$$

Note that  $C$  is a Cantor set since it is perfect, compact, and totally disconnected.

**Theorem 5.2.** *The Cantor set  $C$  from the previous construction has embedding homogeneity group  $\mathbb{Z}$  and is unsplittable.*

*Proof.* It is clear from the construction that each point of  $C - \{w\}$  has local genus 1. Theorem 2.4 applied to  $w$  and the Cantor sets  $C_+ = \overline{\bigcup_{i > 0} C_i}$  and  $C_- = \overline{\bigcup_{i < 0} C_i}$  shows that  $w$  has local genus 2 in  $C$ . Thus any homeomorphism of  $S^3$  that takes  $C$  to  $C$  must fix  $w$ .

Let  $h$  be such a homeomorphism of  $S^3$  taking  $C$  to  $C$ . Let  $T'_i$  be the union of the linked tori in the Antoine chain at the second stage of the construction of  $C_i$ . Let

$$\Lambda_N = \bigcup_{i=-N}^N T_i, \quad \Gamma_N = \bigcup_{i=-N}^N C_i, \quad \text{and} \quad \Lambda'_N = \bigcup_{i=-N}^N T'_i.$$

Fix an integer  $n \in \mathbb{Z}$ . Since  $h(T_n)$  does not contain  $w$ , there is a positive integer  $N_1 > |n|$  such that  $h(C_n) \subset \Gamma_{N_1}$ . Similarly, there is a positive integer  $N_2 > N_1$  such that  $h^{-1}(\Gamma_{N_1}) \subset \Gamma_{N_2}$ .

As in step one in the proof of Theorem 3.1, there is a homeomorphism  $k$  of  $S^3$  to itself, fixed on  $C$ , so that

$$k \left( h \left( \partial(\Lambda_{N_2+1}) \cup \partial(\Lambda'_{N_2+1}) \right) \right) \cap \left( \partial(\Lambda_{N_2+1}) \cup \partial(\Lambda'_{N_2+1}) \right) = \emptyset.$$

Fix a point  $p$  of  $C_n$  and let  $k(h(p)) = h(p) = q \in C_m$ . We will show that  $k(h(C_n)) = h(C_n) = C_m$ . Let  $\ell = k \circ h$ . Since  $\ell(T_n) \cap T_m \neq \emptyset$ , and since the boundaries do not intersect, either  $\ell(T_n) \subset \text{Int}(T_m)$  or  $\text{Int}(\ell(T_n)) \supset T_m$ . We consider these cases separately.

**Case I:**  $\ell(T_n) \subset \text{Int}(T_m)$ . If  $\ell(T_n)$  has geometric index 0 in  $T_m$ , then  $\ell(T_n)$  is contained in a cell in  $T_m$  and so it contracts in  $T_m$ . Since a contraction of  $\ell(T_n)$  meets the boundary of the linked  $\ell(T_{n+1})$ , and since the boundary of  $\ell(T_{n+1})$  is disjoint from the boundary of  $T_m$ ,  $\ell(T_{n+1}) \subset \text{Int}(T_m)$ . Continuing inductively, one finds that one of the following two situations occur when  $\ell(T_n)$  has geometric index 0 in  $T_m$ :

**Case Ia:** All of  $\ell(T_n), \ell(T_{n+1}), \dots, \ell(T_{N_2})$  are contained in  $T_m$  and have geometric index 0 there.

**Case Ib:** There is a  $j, n < j \leq N_2$ , with  $\ell(T_j) \subset \text{Int}(T_m)$  and of geometric index greater than or equal to 1 there.

In Case Ia, it follows that  $\ell(T_{N_2+1}) \subset \text{Int}(T_m)$ . But then  $C_m \subset \Gamma_{N_1}$  and  $h^{-1}(C_m) \cap C_{N_2+1} \neq \emptyset$ , contradicting the choice of  $N_2$ .

In Case Ib, suppose  $\ell(T_j)$  has geometric index  $k$  greater than 1 in  $T_m$ . Then, by Theorem 2.2,  $\ell(T_j)$  cannot be contained in any component of the next stage of the construction contained in  $T_m$  since these have

geometric index 0 in  $T_m$ . So some component of the next stage in  $T_m$  is contained in  $\ell(T_j)$  and has geometric index 0 there by Theorem 2.2. Since the components of the next stage are linked, all components of the next stage in  $T_m$  are contained in  $\ell(T_j)$ . The geometric index of the union of the next stages of in  $T_m$  in  $\ell(T_n)$  is even by Theorem 2.3 and can not be equal to 0. Otherwise, by Theorem 2.2 the union of the next stages of  $T_m$  would have index 0 in  $T_m$ , which is a contradiction. So the geometric index of union of the next stages of in  $T_m$  in  $\ell(T_n)$  is at least 2. Then by Theorem 2.2, the geometric index of union of the next stages of in  $T_m$  in  $T_m$  is at least 4, contradicting the fact that this geometric index is 2.

It follows that  $\ell(T_j)$  has geometric index 1 in  $T_m$  and contains the union of the next stages contained in  $T_m$ . Since  $\ell$  is a homeomorphism that takes  $C$  to  $C$ , it follows from the construction of  $C$  that  $\ell(C_j) = C_m$ . Since  $\ell(p) \in C_m$ ,  $\ell(T_n) \cap \ell(C_j) \neq \emptyset$ , contradicting the fact that  $\ell$  is a homeomorphism.

Thus, neither Case Ia nor Case Ib can occur. So the geometric index of  $\ell(T_n)$  in  $T_m$  must be at least 1. Repeating the argument from Case Ib above, with  $T_j$  replaced by  $T_n$ , we see that  $\ell(T_n)$  has geometric index 1 in  $T_m$  and contains the union of the next stages contained in  $T_m$ . Since  $\ell$  is a homeomorphism that takes  $C$  to  $C$ , it follows from the construction of  $C$  that  $\ell(C_n) = C_m$  as desired.

**Case II:**  $\text{Int}(\ell(T_n)) \supset T_m$ . Then  $\ell^{-1}(T_m) \subset \text{Int}(T_n)$ . The argument from Case I can now be repeated replacing  $\ell$  by  $\ell^{-1}$  and interchanging  $T_n$  and  $T_m$ . It follows that  $\ell^{-1}(C_m) = C_n$  and so  $\ell(C_n) = C_m$  as desired.

Since  $\ell(C_n) = h(C_n) = C_m$ , it must be the case that  $(m - n) \equiv 0 \pmod{3}$ . Continuing as in the proof of the  $\mathbb{Z}_m$  result (Theorem 4.2), we have that for each  $i$ ,  $h(C_i) = C_{i+(m-n)}$ . Recall that for the homeomorphism  $\alpha$  from the construction of  $C$ , it is also the case that

$\alpha^{\frac{m-n}{3}}(C_i) = C_{i+(m-n)}$ . By the rigidity of these Cantor sets, it follows that  $\alpha^{\frac{m-n}{3}}|_{C_i} = h|_{C_i}$ . Thus the embedding homogeneity group of  $C$  is  $\{\alpha^k | k \in \mathbb{Z}\} \simeq \mathbb{Z}$ .

We now show that  $C$  is unsplitable. Assume that  $\Sigma$  is a 2-sphere in  $S^3$  that separates  $C$ . Choose  $\epsilon > 0$  so that the distance from  $\Sigma$  to  $C$  is greater than  $\epsilon$ . Choose  $N$  so that each  $T_i, |i| \geq N$ , has diameter less than  $\epsilon/6$  and is within  $\epsilon/6$  of  $w$ . Since  $\Sigma$  separates  $C$ ,  $w \cup \left( \bigcup_{|i| \geq N} T_i \right)$  must be in one component of  $S^3 - \Sigma$  and there must be points of  $C$  in the other component of  $S^3 - \Sigma$ . So  $\bigcup_{|i| \leq N} T_i$  contains points in both components of  $S^3 - \Sigma$ .

Form an Antoine Cantor set  $C'$  related to  $C$  as follows. Use  $\bigcup_{|i| \leq N} T_i$  as a part of the first stage of the construction. Complete the first stage of the construction by adding an unknotted solid torus  $T$ , linked to  $T_N$  and  $T_{-N}$ , that is within the  $\epsilon/3$ -neighborhood of  $w$ . For successive stages of the Antoine Cantor set  $C'$  in  $T_i, |i| \leq N$ , use the successive stages in forming the Cantor set  $C_i \subset C$ . For successive stages of the Antoine Cantor set  $C'$  in  $T$ , use any Antoine construction.

By construction and the properties of  $\Sigma$ , the 2-sphere  $\Sigma$  separates the Antoine Cantor set  $C'$ , contradicting Remark 3.2.  $\square$

## 6. MAIN RESULTS

Given a finitely generated abelian group  $G$ , we use the results from the previous two sections to construct a unsplitable Cantor set  $C_G$  in  $S^3$  with embedding homogeneity group  $G$ .

**Construction 6.1.** Let  $G \simeq \mathbb{Z}^n \oplus \mathbb{Z}_{m_1} \oplus \mathbb{Z}_{m_2} \cdots \oplus \mathbb{Z}_{m_k}$  be any finitely generated abelian group. Form a simple chain of  $n+k$  pairwise disjoint unknotted solid tori. Figure 4 illustrates the case  $n+k=6$ . Label the tori as  $T_1, T_2, \dots, T_{n+k}$  so that  $T_1$  is only linked with  $T_2$ ,  $T_{n+k}$  is only

linked with  $T_{n+k-1}$ , and so that  $T_i$ ,  $2 \leq i \leq n+k-1$ , is linked with  $T_{i-1}$  and  $T_{i+1}$ .

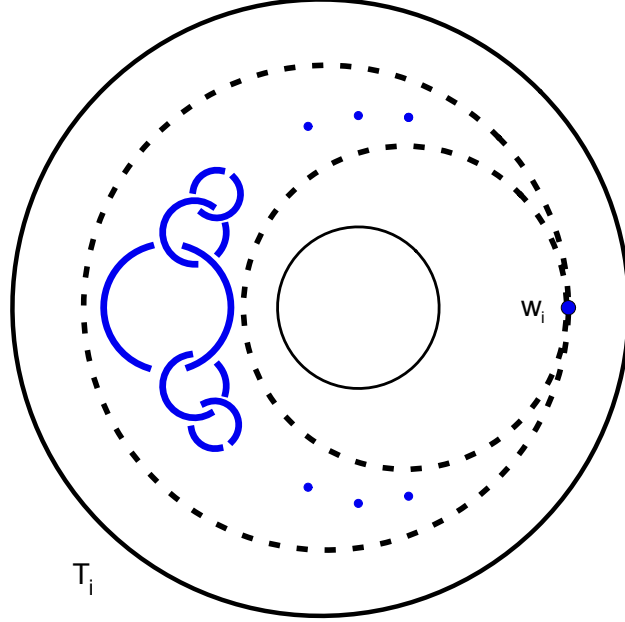


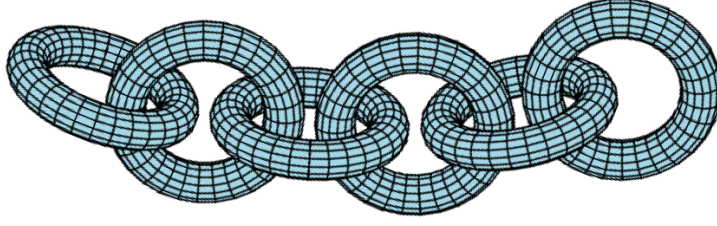
FIGURE 3. Pinched Torus in  $T_i$

For  $1 \leq i \leq n$ , perform Construction 5.1 in  $T_i$ , treating a pinched version of  $T_i$  in the interior of  $T_i$  as the torus  $S_0$  in Construction 5.1. Let  $w_i$  be the limit point corresponding to  $w$  in Construction 5.1. This yields a Cantor set  $C_i$  in  $T_i$  with embedding homogeneity group  $\mathbb{Z}$ . See Figure 3.

For  $n+1 \leq i \leq n+k$ , perform Construction 4.1 for the group  $\mathbb{Z}_{m_{i-n}}$  in  $T_i$ . This yields a Cantor set  $C_i$  in  $T_i$  with embedding homogeneity group  $\mathbb{Z}_{m_{i-n}}$ . Choose all the rigid Cantor sets from Constructions 5.1 and 4.1 to be inequivalent.

Let

$$C_G = \bigcup_{i=1}^{n+k} C_i.$$

FIGURE 4. Example of Antoine Chain Containing  $C_G$ 

**Theorem 6.2.** *The Cantor set  $C_G$  constructed above has embedding homogeneity group  $G$  and is unsplittable.*

*Proof.* For each  $i, 1 \leq i \leq n+k$ , let  $h_i$  be a self-homeomorphism of  $S^3$ , fixed on the complement of  $T_i$ , such that  $h_i|_{C_i}$  generates the embedding homeomorphism group of  $C_i$  ( $\mathbb{Z}$  for  $1 \leq i \leq n$  and  $\mathbb{Z}_{m_i-n}$  for  $n+1 \leq i \leq n+k$ ). Then:

$$\left\{ \left( h_1^{j_1} \circ h_2^{j_2} \dots \circ h_{n+k}^{j_{n+k}} \right) \Big|_{C_G} \right\} \simeq G \simeq \mathbb{Z}^n \oplus \mathbb{Z}_{m_1} \oplus \mathbb{Z}_{m_2} \dots \oplus \mathbb{Z}_{m_k}.$$

Let  $h$  be a homeomorphism of  $S^3$  to itself taking  $C_G$  to  $C_G$ . We will show that  $h|_{C_G} = \left( h_1^{j_1} \circ h_2^{j_2} \dots \circ h_{n+k}^{j_{n+k}} \right) \Big|_{C_G}$  for some choice of  $j_i$ .

**Step 1:** The homeomorphism  $h$  must take each  $C_i$  to itself. As in the proof of Theorem 5.2, there are exactly  $n$  points of genus 2 in  $C_G$ , one in each  $C_i$ ,  $1 \leq i \leq n$ . These are the points  $\{w_1, w_2, \dots, w_n\}$ . The homeomorphism must take this set of genus 2 points to itself.

Let  $T$  be one of the solid torus components of the first stage of the Antoine construction for some  $C_i$ ,  $1 \leq i \leq n+k$ . As in the proofs of Theorems 4.2 and 5.2, after a general position adjustment,  $h(T)$  must either lie in the interior of some solid torus component  $T'$  of the first stage of the Antoine construction for some  $C_j$ , or  $T'$  must lie in the interior of  $h(T)$ . A similar argument to that for Theorem 5.2 shows that  $N(h(T), T') = 1$  or  $N(T', h(T)) = 1$  and that  $h(C_i \cap T) = C_j \cap T'$ .



This same argument can be applied to all first stage tori in  $C_i$ , resulting in the fact that  $h(C_i) = C_j$ . Because of the inequivalence of the rigid Cantor sets used in the construction,  $i = j$  and  $h(C_i) = C_i$ .

**Step 2:** For each  $i$ ,  $h|_{C_i} = h_i^{k(i)}|_{C_i}$  for some  $k(i)$ . By Step 1, we have that  $h(C_i) = C_i$ . It follows from the construction that  $h|_{C_i} = h_i^{k(i)}$  for some  $k(i)$ . From this, it follows that  $h|_{C_G} = \left( h_1^{j_1} \circ h_2^{j_2} \dots \circ h_{n+k}^{j_{n+k}} \right)|_{C_G}$  for some choice of  $j_i$ .

Thus, the embedding homeomorphism group of  $C_G$  is isomorphic to  $G$ .

**Step 3:**  $C_G$  is unsplittable. Let  $\Sigma$  be a 2-sphere in  $S^3$  separating  $C_G$ . As in the proof of Theorem 5.2, an Antoine Cantor set with first stage  $\bigcup_{i=1}^{n+k} T_i$  can be formed so that  $\Sigma$  separates this Antoine Cantor set. This is a contradiction. See Remark 3.2.  $\square$

**Corollary 6.3.** *Let  $G \simeq \mathbb{Z}^n \oplus \mathbb{Z}_{m_1} \oplus \mathbb{Z}_{m_2} \cdots \oplus \mathbb{Z}_{m_k}$  be any finitely generated abelian group. Then there is a irreducible open 3-manifold  $M_G$  whose Freudenthal compactification is  $S^3$  with the following properties:*

- *the end set is homeomorphic to a Cantor set,*
- *the end homogeneity group of  $M_G$  is isomorphic to  $G$ , and*
- *$M_G$  is genus one at infinity except for  $n$  ends where it is genus two at infinity.*

*Proof.* Let  $M_G$  be  $S^3 - C_G$  where  $C_G$  is as in Construction 6.1. Then the end set of  $M_G$  is  $C_G$  and the end homogeneity group of  $M_G$  is isomorphic to the embedding homogeneity group of  $C_G$ .  $M_G$  is irreducible because  $C_G$  is unsplittable. The first two claims now follow from Theorem 6.2. The third claim follows from the proof of this theorem.  $\square$

**Remark 6.4.** Note that for each finitely generated abelian group  $G$  as above, there are uncountably many non-homeomorphic 3-manifolds as in the corollary. This follows from varying the rigid Cantor sets used in the construction.

## 7. QUESTIONS

The following questions arise from a consideration of the results in this paper.

**Question 7.1.** *If a finitely generated abelian group is infinite, is there an open 3-manifold with end homogeneity group  $G$  that is genus one at infinity?*

**Question 7.2.** *Given a finitely generated abelian group  $G$ , are there simply connected open 3-manifolds with end homogeneity group  $G$ ?*

**Question 7.3.** *Is the mapping class group of the open 3-manifold  $M_G$  isomorphic to  $G$ ?*

**Question 7.4.** *If  $G$  is a finitely generated non-Abelian group, is there an open 3-manifold with end homogeneity group  $G$ ?*

**Question 7.5.** *If  $G$  is a non-finitely generated group, is there an open 3-manifold with end homogeneity group  $G$ ?*

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