DISTINGUISHING BING-WHITEHEAD CANTOR SETS

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ABSTRACT. Bing-Whitehead Cantor sets were introduced by De-Gryse and Osborne in dimension three and greater to produce examples of Cantor sets that were non standard (wild), but still had simply connected complement. In contrast to an earlier example of Kirkor, the construction techniques could be generalized to dimensions bigger than three. These Cantor sets in S^3 are constructed by using Bing or Whitehead links as stages in defining sequences. Ancel and Starbird, and separately Wright characterized the number of Bing links needed in such constructions so as to produce Cantor sets. However it was unknown whether varying the number of Bing and Whitehead links in the construction would produce non equivalent Cantor sets. Using a generalization of geometric index, and a careful analysis of three dimensional intersection patterns, we prove that Bing-Whitehead Cantor sets are equivalently embedded in S^3 if and only if their defining sequences differ by some finite number of Whitehead constructions. As a consequence, there are uncountably many non equivalent such Cantor sets in S^3 constructed with genus one tori and with simply connected complement.

1. BACKGROUND

Two Cantor sets X and Y in S^3 are *equivalent* if there is a self homeomorphism of S^3 taking X to Y. If there is no such homeomorphism, the Cantor sets are said to be *inequivalent*, or *inequivalently embedded*.

There has been an extensive study in the literature of non standard Cantor sets in S^3 (those that are not equivalent to the standard middle thirds Cantor set). Recent interest is partly due to the fact that such Cantor sets are often the invariant sets of certain dynamical systems. See [BC87, GRŽ05].

Antoine [Ant20] constructed the first example of a non standardly embedded Cantor set. Sher [She68] showed that there were uncountably many inequivalent Cantor sets in S^3 by varying the number of components in the Antoine construction. These Cantor sets all had non simply connected complement and so were non standard.

Kirkor [Kir58] constructed the first non standard Cantor set in R^3 with simply connected complement. Any Cantor set in R^3 with simply connected complement has the property that any 2 points in the Cantor set can be separated by a 2-sphere missing the Cantor set (see [Sko86]). This allows the components of the stages of a defining sequence to be separated and makes the non equivalence to the standard Cantor set much more difficult to detect. DeGryse and Osborne [DO74] used a generalization of the Bing-Whitehead construction to produce non standard Cantor sets with simply connected complement in all dimensions greater than or equal to three.

Ancel and Starbird [AS89], and Wright [Wri89] analyzed exactly which Bing-Whitehead constructions yielded Cantor sets. It was unknown whether changing the number of Bing and Whitehead links in the construction would yield inequivalent Cantor sets. Željko [Žel00] in his dissertation conjectured that if two Bing-Whitehead constructions yielded equivalent Cantor sets, then the constructions differed in a finite number of Whitehead construction. This is essentially Question 7 in [GR07]. In this paper, we prove that this conjecture is true.

See [Shi74]), [Bla51], [Žel05], [Žel01], [GRŽ05], and the bibliography in [GR07] for additional examples of non standard Cantor sets. Robert Myers [Mye88] has a very interesting paper on contractible 3-manifolds that use techniques very similar to the ones used in this paper even though there are no Cantor sets mentioned

In the next section we list the terminology and notation that we use and list the properties of Bing and Whitehead links from Wright's paper [Wri89] that are needed in our analysis. We also list the main result that we obtain. In Section 3, we list the results on geometric linking and geometric index that we need. The results in this section follow from a generalization of Schubert's [Sch53] results to links with more that one component. In Section 4, we prove that the boundaries of the stages in the construction for a Bing-Whitehead compactum can be made disjoint from boundaries of another defining sequence for the same compactum. In Section 5 we prove the main result. We end with some additional questions.

2. Properties of Bing and Whitehead links

Bing and Whitehead Links. We work in the piece-wise linear category. A *link* is the finite union of disjoint simple closed curves. A torus is a 2-manifold homeomorphic to the product of two simple closed curves. A solid torus is a 3-manifold homeomorphic to a disk cross a simple closed curve. We denote the interior and boundary of a manifold M by Int M and ∂M, respectively. Let T be a solid torus. Throughout this paper, we assume that the tori we are working with are unknotted in S^3. (The results and constructions also work in R^3.) A *Bing link* in T is a union of 2 linked tori $F_1 \cup F_2$ embedded in T as shown in Figure 1. A *Whitehead link* in T is a torus W embedded in T as shown in the Figure. For background details and terminology, see Wright's paper [Wri89]. The link terminolgy arises from the link consisting of the cores of the interior tori together with a meridional curve on the outer torus.
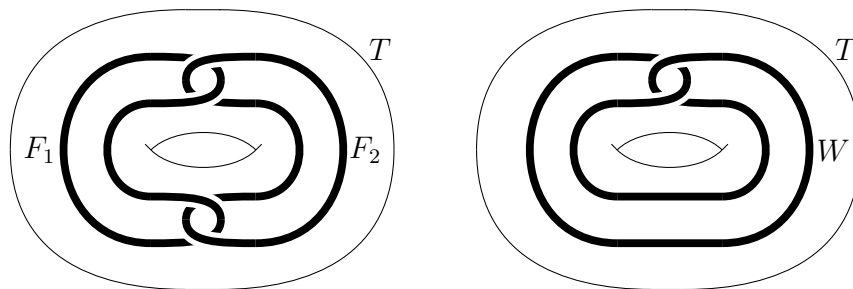

FIGURE 1. Bing and Whitehead Constructions

Construction of Bing-Whitehead Compacta. For completeness and consistency of notation, we outline the steps in the construction of Bing-Whitehead compacta. Let M_0 be an unknotted torus in S^3, and M_1 be obtained from M_0 by placing a Bing construction in M_0. Inductively obtain M_k from M_{k-1} by placing a Bing construction in each component of M_{k-1} or by placing a Whitehead construction in each component of M_{k-1}. Let n_1 be the number of consecutive Bing stages in the construction before the first Whitehead stage, and let n_k be the number of consecutive Bing stages placed between the $(k-1)$st and kth Whitehead stages of M.

Definition 2.1. *The Bing-Whitehead compactum associated with this construction is defined to be*

$$X = \bigcap_{i=1}^{\infty} M_i \text{ and is denoted } X = BW(n_1, n_2, \ldots)$$

We also define $M_i, i < 0$ so that M_i is a Whitehead construction in M_{i-1} and let X^∞ be $\bigcap_i (S^3 \setminus M_i)$. X^∞ is called the compactum at infinity associated with X. We assume that infinitely many of the $M_i, i > 0$, arise from Bing constructions and that infinitely many of them arise from Whitehead constructions.

It is known [AS89, Wri89] that this construction can be done so as to yield a Cantor set if and only if the series $\sum_i n_i 2^{-i}$ diverges. Specifically, if G is the decomposition of S^3 consisting of the components of $X = BW(n_1, n_2, \ldots)$ and the remaining points of S^3, then S^3/G is homeomorphic to S^3 if and only if this condition holds. The image of X under the quotient map is then a Cantor set in S^3 called a Bing-Whitehead Cantor Set. Standard results from decomposition theory [Dav86] then imply that in this case, the construction of X can be done so that the components of X are points and thus X itself is a Cantor set.

We introduce one additional definition that will be needed in the proof of the main theorem in Section 5.

Definition 2.2. *Suppose X is a BW compactum with defining sequence $(M_i), i \geq 0$. The **BW pattern** for X with respect to (M_i) is the sequence $(\alpha_1, \alpha_2, \alpha_3, \ldots)$ where $\alpha_i = 1$ if M_i is obtained from M_{i-1} by placing a Whitehead construction in each component, and where $\alpha_i = 2$ if M_i is obtained from M_{i-1} by placing a Bing construction in each component.*

Geometric properties. We list the key results from Wright's paper that will be needed in what follows.

Lemma 2.3. *Let M be a Bing or Whitehead link in a solid torus T.*

- *[Wri89, Lemma 4.1] $T - \text{Int } M$ is boundary incompressible , i.e. there is no 2-disk $D \subset T - \text{Int } M$ such that $D \cap (\partial T \cup \partial M) = \partial D$ with ∂D essential in $\partial T \cup \partial M$.*
- *[Wri89, Lemma 4.2] There is no annulus inside T connecting essential loops on two different components of $\partial M \cup \partial T$.*

Lemma 2.4. *Let X be a Bing-Whitehead compactum and X^∞ the associated continuum at infinity.*

- [Wri89, Theorem 4.6] *No sphere in the complement of $X \cup X^\infty$ separates $X \cup X^\infty$.*
- [Wri89, Theorem 4.3] *A loop on the boundary of M_i is essential in the boundary of M_i if and only if it is essential in the complement of $X \cup X^\infty$.*
- [Wri89, Theorem 4.4] *If loops ℓ_1 and ℓ_2 in ∂M_i and ∂M_j respectively, $i \neq j$, are homotopic in the complement of $X \cup X^\infty$, then they are inessential in $X \cup X^\infty$.*

Main Result. Our ultimate goal is to determine when two Bing-Whitehead constructions (M_i) and (N_j) yield Cantor sets X_1 and X_2 that are equivalently embedded.

Theorem 2.5 (Main Theorem). *Let X_1 be a Bing-Whitehead Cantor set associated with a defining sequence (M_i) and let X_2 be a Bing-Whitehead Cantor set associated with a defining sequence (N_j). If X_1 and X_2 are equivalently embedded, then the defining sequences differ in a finite number of Whitehead constructions. Specifically, if $X_1 = BW(m_1, m_2, \ldots)$ with respect to M_i and $X_2 = BW(n_1, n_2, \ldots)$ with respect to N_j, then there are integers p and q such that $\sum_{i=1}^{p} m_i = \sum_{j=1}^{q} n_j$ and $m_{p+k} = n_{q+k}$ for all $k \geq 1$.*

Remark 2.6. *Note that the converse of Theorem 2.5 is also true. This was also observed in Željko's dissertation [Žel00]. Assume there are integers p and q such that $\sum_{i=1}^{p} m_i = \sum_{j=1}^{q} n_j$ and $m_{p+k} = n_{q+k}$ for all $k \geq 1$. Then there are homeomorphisms of h_1 and h_2 of S^3 taking M_p and N_q onto a collection of $2^{\sum_{i=1}^{p} m_i}$ pairwise disjoint, unknotted and unlinked tori. Using the fact that $m_{p+k} = n_{q+k}$ for all k, one can construct inductively homeomorphisms that take the components of (N_{q+k}) onto the components of $h_1(M_{p+k})$. Because X_1 and X_2 are Cantor sets, these homeomorphisms can be chosen so that the limit is a homeomorphism of S^3 to itself taking X_2 to $h_1(X_1)$*

Corollary 2.7. *There are uncountably many inequivalent Bing-Whitehead Cantor sets in S^3.*

Proof. To get uncountably many distinct examples, start with the example

$$BW(1, 2, 4, \ldots, 2^i, 2^{i+1}, \ldots)$$

Let $\alpha = (j_0, j_1, j_2, \ldots)$ be an increasing sequence of positive integers. The examples we seek are of the form

$$BW(1 + 3^{j_0}, 2 + 3^{j_1}, 4 + 3^{j_2}, \ldots, 2^i + 3^{j_i}, 2^{i+1} + 3^{j_{i+1}}, \ldots)$$

By Theorem 2.5, for distinct sequences of increasing integers, no two of these are equivalent. □

3. Algebraic and Geometric Index

Algebraic Index. If S is a solid torus in another solid torus T, the *algebraic index* of S in T is $|\alpha|$ where α is the integer in $H_1(T)$ represented by the center line of S. Algebraic index is multiplicative, so that if $S \subset T \subset U$ are solid tori, the algebraic index of S in U is the product of the algebraic index of S in T with the algebraic index of T in U. Note that the algebraic index of a Whitehead link in the torus containing it is 0, as is the algebraic index of each component of a Bing link.

Geometric Index. If K is a link in the interior of a solid torus T, then we denote the *geometric index* of K in T by $N(K, T)$. The geometric index is the minimum of $|K \cap D|$ over all meridional disks D of T. A *core* of a solid torus T in 3-space is a simple closed curve J so that T is a regular neighborhood of J. Likewise, a core for a finite union of disjoint solid tori is a link consisting of one core from each of the solid tori. If T is a solid torus and M is a finite union of disjoint solid tori so that $M \subset \text{Int } T$, then the geometric index $N(M, T)$ of M in T is $N(K, T)$ where K is a core of M. The geometric index of a Bing link $F_1 \cup F_2$ in a torus T is 2. The geometric index of a Whitehead link W in a torus T is also 2.

Theorem 3.1. *Let T_0 and T_1 be unknotted solid tori in S^3 with $T_0 \subset \text{Int } T_1$ and $N(T_0, T_1) = 1$. Then ∂T_0 and ∂T_1 are parallel; i.e., the manifold $T_1 - \text{Int } T_0$ is homeomorphic to $\partial T_0 \times I$ where I is the closed unit interval $[0, 1]$.*

Proof. The proof follows from work of Schubert [Sch53] and regular neighborhood theory. Let J be a core of T_0. Since T_0 is unknotted, J is an unknotted simple closed curve. The geometric index of J in T_1 is one. By Schubert, J is either a core of T_1 or a sum of knots with a core. Since J is unknotted, J must be a core of T_1. Since J is a core of both T_0 and T_1, regular neighborhood theory [RS72] shows that $T_1 - \text{Int } T_0$ is homeomorphic to $\partial T_0 \times I$. □

Theorem 3.2. *Let T_0 be a finite union of disjoint solid tori. Let T_1 and T_2 be solid tori so that $T_0 \subset \text{Int } T_1$ and $T_1 \subset \text{Int } T_2$. Then $N(T_0, T_2) = N(T_0, T_1) \cdot N(T_1, T_2)$.*

Proof. Schubert proves the case where T_0 is a single solid torus, but his proof works for the above case with no changes. $\square$

There is one additional result we will need in Section 4.

Theorem 3.3. *Let T be a solid torus in S^3 and let T_1, T_2 be unknotted solid tori in T, each of geometric index 0 in T. Then the geometric index of $\cup_{i=1}^{2} T_i$ in T is even.*

Proof. If the geometric index were odd, then there is a meridional disk D of T that intersects the cores of $T_1 \cup T_2$ transversally an odd number of times. So this means that D must intersect the core of either T_1 or T_2 an odd number of times. But if a meridional disk of T intersects a simple closed curve J transversally an odd number of times, the algebraic index of J in T is odd and so J is essential in T. However, the cores of the T_i are both inessential because they lie in a ball in T. $\square$

Boundary Parallel Tori. The next three results make use of the material on geometric index to determine when the boundaries of certain tori are parallel. These results are used in the proof of the main theorem in Section 5 to inductively match up stages in different Bing-Whitehead defining sequences.

Theorem 3.4. *Let W be a Whitehead link in the solid torus T in S^3. If $T' \subset T$ is a solid unknotted torus whose boundary separates ∂W from ∂T, then $\partial T'$ is parallel to either ∂W or ∂T.*

Proof. Since $\partial T'$ separates ∂W from ∂T, we have $W \subset \operatorname{Int} T'$ and $T' \subset \operatorname{Int} T$. Since $\mathrm{N}(W, T') \cdot \mathrm{N}(T', T) = \mathrm{N}(W, T) = 2$, either $\mathrm{N}(W, T') = 1$ or $\mathrm{N}(T', T) = 1$. The conclusion now follows from Theorem 3.2. $\square$

Theorem 3.5. *Let $F_1 \cup F_2$ be a Bing link in a solid torus T in S^3. If $T' \subset T$ is a solid unknotted torus whose boundary separates $\partial(F_1 \cup F_2)$ from ∂T, then $\partial T'$ is parallel to ∂T.*

Proof. Since $\partial T'$ separates $\partial(F_1 \cup F_2)$ from ∂T, we have $F_1 \cup F_2 \subset \operatorname{Int} T'$ and $T' \subset \operatorname{Int} T$. Since $\mathrm{N}(F_1 \cup F_2, T') \cdot \mathrm{N}(T', T) = \mathrm{N}(F_1 \cup F_2, T) = 2$, either $\mathrm{N}(F_1 \cup F_2, T') = 1$ or $\mathrm{N}(T', T) = 1$. We show $\mathrm{N}(F_1 \cup F_2, T') = 1$ is impossible. Suppose $\mathrm{N}(F_1 \cup F_2, T') = 1$, then $\mathrm{N}(F_i, T') = 1$ for either $i = 1$ or $i = 2$. Now $0 = \mathrm{N}(F_i, T) = \mathrm{N}(F_i, T') \cdot \mathrm{N}(T', T) = \mathrm{N}(T', T) \neq 0$, a contradiction. So we conclude that $\mathrm{N}(T', T) = 1$ and the conclusion now follows from Theorem 3.2. $\square$

Theorem 3.6. *Let $F_1 \cup F_2$ be a Bing link in the solid torus T in S^3. If S is the boundary of a solid unknotted torus that separates $\partial F_1 \cup \partial F_2 \cup \partial T$, then S is parallel to one of ∂F_1, ∂F_2, ∂T.*

Proof. If S separates ∂T from $\partial F_1 \cup \partial F_2$, then we can invoke the previous theorem. The other cases follow from the fact that there are homeomorphisms of S^3 to itself that take $T - \text{Int}(F_1 \cup F_2)$ to itself and take ∂F_i to ∂T. These homeomorphisms follow from the (well known) fact that $F_1 \cup F_2 \cup (S^3 - \text{Int } T)$ are Borromean Rings. ☐

4. Boundary Intersections of Defining Sequences

Setup. For the rest of this section, we assume that there is a Bing-Whitehead compactum X with two defining sequences (M_k) and (N_k). Let X_M^∞ be the continuum at infinity associated with the first defining sequence and let X_N^∞ be the continuum at infinity associated with the second defining sequence.

Theorem 4.1. *Let $X, (M_k), (N_k), X_N^\infty,$ and X_M^∞ be as above. Suppose that i and j are chosen so that*

- *M_{i-1} is in N_1 and so is in the complement of X_N^∞*
- *N_{j-1} is in M_1 and so is in the complement of X_M^∞*

Let n be a fixed integer. Then there is a homeomorphism h of S^3 to itself, fixed on $X \cup (S^3 - M_1) \cup (S^3 - N_1)$, so that $h(\partial(M_{i+m})) \cap \partial(N_{j+\ell}) = \emptyset$ for each nonnegative m and ℓ less than n.

The remainder of this section is devoted to the proof of this Theorem. We will need to apply the following Lemmas. Note that Lemma 4.2 is the case $n = 0$ of Theorem 4.1.

Lemma 4.2. *Let $X, (M_k), (N_k), X_N^\infty,$ and X_M^∞ be as above. Suppose that i and j are chosen so that*

- *M_{i-1} is in N_1 and so is in the complement of X_N^∞*
- *N_{j-1} is in M_1 and so is in the complement of X_M^∞*

Then there is a homeomorphism h of S^3 to itself, fixed on $X \cup (S^3 - M_1) \cup (S^3 - N_1)$, so that $h(\partial M_i) \cap \partial N_j = \emptyset$.

Lemma 4.3. *Let $X, (M_k), (N_k), X_N^\infty,$ and X_M^∞ be as above. Suppose:*

- *T' is a component of N_j and N_j is in the complement of X_M^∞.*
- *$M_i \cap T' \subset \text{Int}(T')$ and consists of components $T_1, \ldots T_r$ of M_i*

Then there is a self homeomorphism h of S^3, fixed on $X \cup (S^3 - T')$, so that $h(\partial(\cup_{k=1}^r T_k)) \cap \partial(N_{j+1}) = \emptyset$.

Lemma 4.4. *Let* $X, (M_k), (N_k), X_N^\infty$, *and* X_M^∞ *be as above. Suppose:*

- T *is a component of* M_i *and* M_i *is in the complement of* X_N^∞.
- $N_j \cap T \subset \text{Int}(T)$ *and consists of components* $T_1', \dots T_r'$ *of* N_j

Then there is a self homeomorphism h *of* S^3, *fixed on* $X \cup (S^3 - T)$, *so that* $h(\partial(M_{i+1})) \cap \partial(\cup_{k=1}^r T_k') = \emptyset$.

Proof of Lemma 4.2. Adjust the components of ∂M_i, ∂N_{j-1}, ∂N_j, and ∂N_{j+1} so that they are in general position. This implies that the boundaries of these components intersect in a finite collection of pairwise disjoint simple closed curves. We will successively remove these curves of intersection by homeomorphisms of S^3.

Removing Trivial Curves of Intersection.

Focus on one component T of M_i. Consider $\partial T \cap \partial N_j$. This intersection, if nonempty, consists of a finite number of simple closed curves. By Lemma 2.4, and by the hypotheses of Lemma 4.2, one of these curves is inessential on ∂T if and only if it is inessential on some component of ∂N_j. If there are any inessential curves, choose a component T' of N_j that contains one in $\partial T'$. Choose an innermost inessential simple closed curve α on $\partial T'$. Since α is innermost, it bounds a disk D' with interior missing ∂T. The curve α also bounds a disk D in ∂T.

The 2-sphere $D \cup D'$ bounds a three-cell in $M_1 \cap N_1$ that by Lemma 2.4 contains no points of X. Use this three-cell to push D onto D' and then a little past D' into an exterior collar on the cell by a homeomorphism h of S^3. This homeomorphism can be chosen to fix X, $S^3 - M_1$, and $S^3 - N_1$. This has the result that $h(\partial T) \cap \partial T'$ has fewer curves of intersection than $\partial T \cap \partial T'$, and so that no new curves of intersection with $\partial(N_j)$ are introduced. Continuing this process eventually removes all inessential curves of intersection on ∂T. Repeating this process for each component of M_i removes all inessential curves of intersection of the boundaries of M_i and N_j. Repeating the process with N_{j-1} and N_{j+1} completes the first step of the proof.

So there is a homeomorphism h_1 of S^3 to itself, fixed on $X \cup X_M^\infty \cup X_N^\infty$ such that $h_1(\partial M_i) \cap (\partial N_{j-1} \cup \partial N_j \cup \partial N_{j+1})$ has no nontrivial curves of intersection. To simplify notation in what remains, we will refer to $h_1(M_i)$ as (the new) M_i.

Remark 4.5. *At this point, let S be a component of M_i. Then there is at most one component S' of N_j for which $\partial S \cap \partial S' \neq \emptyset$, and if this is the case, then $\partial S \cap \partial N_{j-1} = \emptyset$ and $\partial S \cap \partial N_{j+1} = \emptyset$. This follows directly from Lemma 2.3. In fact, the curves of intersection on ∂S must be parallel (p, q) torus curves and the corresponding curves on $\partial S'$ must be parallel (s, t) curves. If both p and q are greater than 1, so that the torus curve is a nontrivial knot, then $(s, t) = (p, q)$ or $(s, t) = (q, p)$ by results from Rolfsen* [Rol76], *but we do not use this observation.*

We now work towards removing these remaining curves of intersection of the boundaries, so that the components of (M_i) under consideration either are contained in or contain the components of (N_j) under consideration. Consider an annulus A on the boundary of S bounded by two adjacent curves of the intersection of ∂S and $\partial S'$. Choose this annulus so that its interior lies in the interior of S'. We consider the separate possibilities for how the boundary curves of A lie on S'.

Curves of intersection on S' that are (p, q) curves for $p \geq 2$.

Consider a meridional disc D for S' in general position with respect to A so that $D \cap A$ consists of p arcs intersecting the boundary of D in endpoints and of simple closed curves. Figure 2 illustrates the situation when $p = 5$ and $q = 3$. The shaded regions indicate the intersection of the next stage N_{j+1} with D.
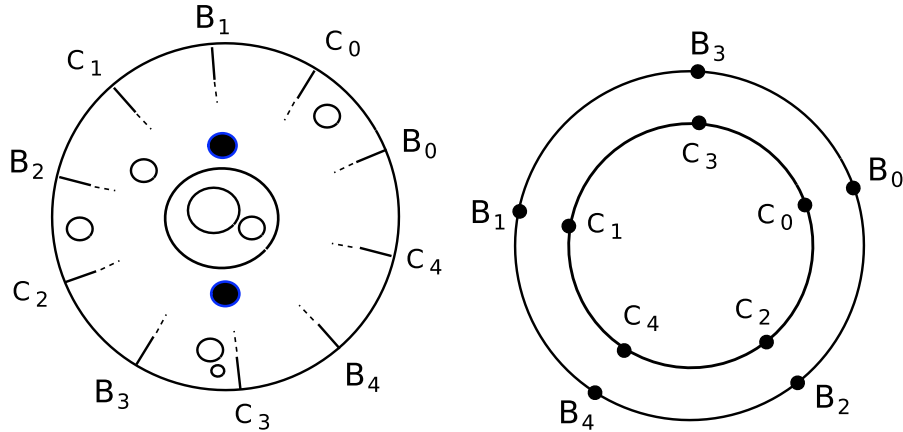

FIGURE 2. Meridional Disc D of S' and Annulus A

Label the boundary curves of the annulus A as curves B and C. Label the intersection points of B with the meridional disc D sequentially around the boundary of D as $B_0, B_1, \ldots B_{p-1}$ and similarly label

the intersection points of C with D as $C_0, C_1, \ldots C_{p-1}$. Because B and C are parallel (p, q) curves on the boundary of S', the intersection points B_i and C_i must alternate. We have not yet indicated how the arcs leaving the points C_i and B_i are connected.

The corresponding points on the annulus A are labeled sequentially along the curve B as $B_0, B_q, B_{2q}, \ldots B_{(p-1)q}$ where subscripts are taken mod p. The points on the annulus A along the C curve are similarly labeled sequentially $C_0, C_q, C_{2q}, \ldots C_{(p-1)q}$. Again, Figure 2 illustrates the case $p = 5$ and $q = 3$.

We will argue that the intersection of A with D can be adjusted using cut and paste techniques so that the end result is intersections as in one of the two cases in Figure 3.
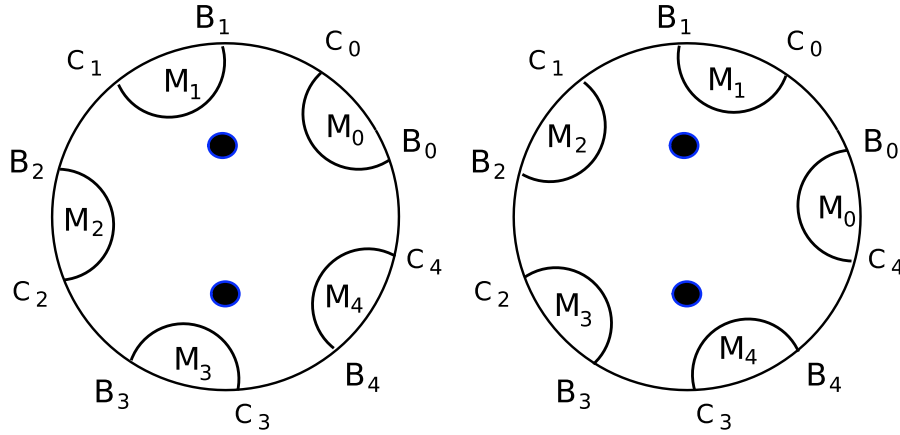

FIGURE 3. Meridional Disc D of S' after Adjustment

Each of the regions labeled M_i will be shown to be a meridional disc of a solid torus that is contained in S'. This solid torus will then be used to push across and remove the intersections of A with D.

Refer back to Figure 2. As a first step, in adjusting the intersection of D and A we show how to remove simple closed curves of intersection. Each simple closed curve is trivial in A, otherwise a (p, q) curve for $p \geq 2$ on the boundary of S' would be null homotopic in S'. None of the simple closed curves can enclose either or both of the shaded regions indicated because they are contractible in A and thus contractible in S' missing X. Choosing an innermost such simple closed curve in D, the intersection can be removed by an argument similar to that used in removing trivial curves of intersection in the previous section. Specifically, there is a homeomorphism h from S^3 to itself, fixed on X

and the complement of S' such that $h(A) \cap D$ has fewer simple closed curves of intersection than $A \cap D$ does. Inductively, all such simple closed curves of intersection can be removed by a self homeomorphism of S^3 fixed on X and on $S^3 - S'$.

After such simple closed curves of intersection are removed, we are left with the situation pictured in Figure 4. Again, we have not yet indicated how the arcs emanating form the boundary points are connected.

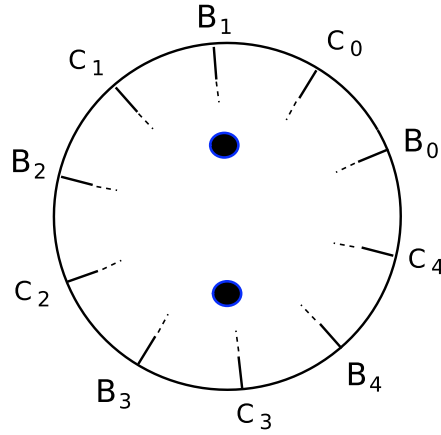

FIGURE 4. Disc D of S' after Simple Closed Curves Removed

First note that if any B_i were joined to a B_j, the arc joining them would separate the disc D and leave an odd number of boundary points on both sides. Since the boundary points are joined in pairs, this is not possible. So each B_i is joined to some C_j by an arc of intersection of A with D.

Next, consider these arcs in the annulus A as in Figure 5.

If point B_0 is joined by an arc of intersection to point C_{kq}, then each point B_{iq} must be joined to the point $C_{(k+i)q}$. Otherwise it would not be possible to have disjoint arcs joining the points on B to the points on C.

Now consider these arcs of intersection again in D as in Figure 4. Since the B point with subscript $iq(\bmod \text{ p})$ is joined to the C point with subscript $(i + k)q \pmod{\text{p}}$, the difference in indices of any two of the joined points is $(i + k)q - iq(\bmod \text{ p}) = kq(\bmod \text{ p})$. Unless this difference is 0 or $p - 1$, it is not possible to place the p arcs in D in a pairwise disjoint fashion. Thus either each B_i in D is joined by an
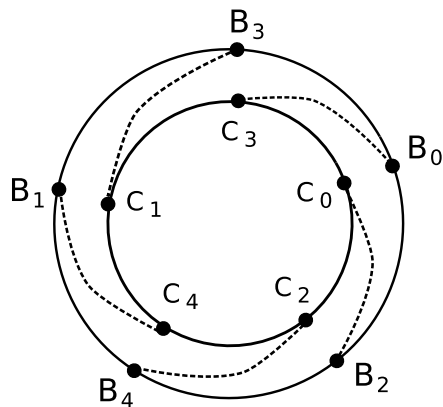
FIGURE 5. Annulus A with Arcs of Intersection

arc to C_i or each is joined by an arc to $C_{i-1 \pmod p}$. Thus, the arcs of intersection are as pictured in Figure 3.

The intersection of the annulus A with $\partial(S')$ separates $\partial(S')$ into two annuli. Let A_1 be the annulus whose intersection with D consists of p arcs joining the same points of the boundary of D as the arcs of intersection of A and D. Then $A \cup A_1 = T_1$ is a torus. See Figure 6 for an illustration of this in one of the cases from Figure 3.
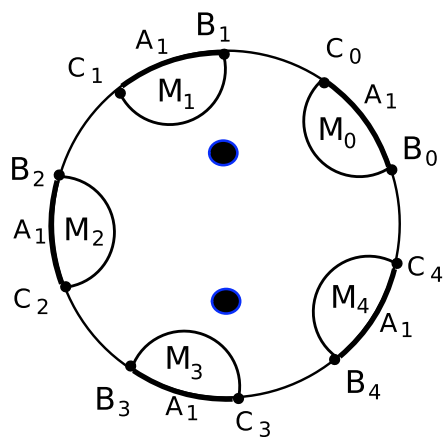
FIGURE 6. D with regions M

Without loss of generality, B_0 is joined to C_0 by an arc α_0 of the intersection of A with D. Let β_0 be the arc in A_1 in the boundary of D joining the endpoints of α_0. The loop $\alpha_0 \cup \beta_0$ is a nontrivial loop in T_1 and T_1 separates S^3 into two components. Let D_1 be the component

that contains the disc M_0 in D bounded by $\alpha_0 \cup \beta_0$. Since $\alpha_0 \cup \beta_0$ bounds a disc in D_1, D_1 is a solid torus by a standard argument. (See [Rol76]).

We now show that the next stage of the construction in S' cannot intersect D_1. Notice that the geometric and algebraic index of D_1 in S' is $p \geq 2$. The geometric index of the next stage of N in S' is 2. If the next stage is a Whitehead construction W in S' that lies in D_1, and the geometric index of W in D_1 is 0 or > 1, there is a contradiction by Theorem 3.2. If the geometric index of W in D_1 is 1, then the algebraic index of W in S' is the same as the algebraic index of D_1 in S' which is $p \neq 0$, again a contradiction.

If the next stage of S' in D is a Bing construction $B = F_1 \cup F_2$ in S', and one component, say F_1, lies in D_1, then the geometric index of F_1 in T_1 must be zero because the geometric index of F_1 in S' is zero. If F_2 does not also lie in D_1, then F_1 lies in a ball that lies in D_1 and hence, misses F_2, a contradiction. If both components of B lie in D_1 then by Theorem 3.3 the geometric index of B in D_1 is even and is thus 0 or > 1. This implies by Theorem 3.2 that the geometric index of B in S' is 0 or ≥ 4, a contradiction.

The intersection of S with S' corresponding to A can now be removed by a homeomorphism of S^3 fixed on X and on the complement of a small neighborhood of S' that takes A through D_1 to an annulus parallel to A_1 and just outside of S'. Inductively, all curves of intersection of S with S' can be removed by a homeomorphism of S^3 fixed on X and the complement of a small neighborhood of S'.

Curves of intersection on S' that are (p, q) curves for $p = 1$.

An argument similar to that in the preceding section can be used. After removing trivial curves of intersection, we are left with an intersection of A with S' as pictured in Figure 7. A divides S' into two tori, labeled U and V in the figure.

The next stage of the construction is either in the solid torus labeled U or in the solid torus labeled V. This is clear if the next stage is a Whitehead construction.

If the next stage of S' in D is a Bing construction $B = F_1 \cup F_2$ in S', and only one component, say F_1, lies in V, then the geometric index of F_1 in V must be zero because the geometric index of F_1 in S' is zero. But then F_1 lies in a ball that lies in V and, hence, misses F_2, a contradiction. So both components of the next stage lie entirely in
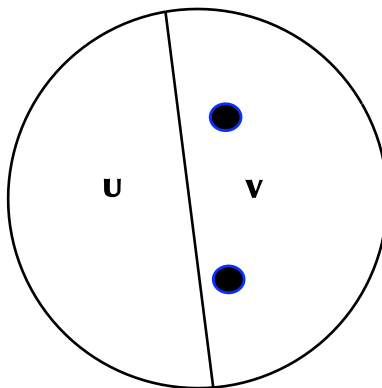
FIGURE 7. The Case $p = 1$

U or entirely in V. The intersection of A with D can be removed by pushing across the other torus.

Curves of intersection on S' that are (p,q) curves for $p = 0$.

In this case the curve is a $(0, q)$ curve for the torus S', but it is a $(q, 0)$ curve for the complementary torus with $q \neq 0$. In this case there is an annulus A on the boundary of S that has its interior in the exterior of S', so that the intersection of A with the boundary of S' consists of curves in the intersection of the boundaries of S and S'. We have essentially turned the problem inside out, and we can use the previous methods to push A to the interior of S' fixed on a slightly shrunken S', all the other components of M_i, and the complement of M_{i-1}.

The discussion above completes the proof of Lemma 4.2 $\qquad\square$

Proof of Lemmas 4.3 and 4.4. The proofs of these two lemmas are virtually identical, with M and N interchanged in the second lemma. For the proof of Lemma 4.3, under the assumption that $M_i \cap T' \subset \text{Int}(T')$ and consists of components $T_1, \dots T_r$ of M_i, one mimics the proof of Lemma 4.2, to make each boundary of T_i disjoint from the boundaries of the one or two components of N_{j+1} in T'. The only additional step is taking care that each homeomorphism from the proof of Lemma 4.2 can be achieved fixing $S^3 - T'$. This is clear because the 3-cells or tori used as guides for these homeomorphisms are all in the interior of T' and all miss X.

Remark 4.6. *Note that the hypotheses of Lemma 4.3 give that the components of M_i intersecting T' in N_j are all interior to T', and*

so their boundaries miss the boundary of T' and thus the boundary of N_j. After the homeomorphism of the Lemma, the boundaries of the components of M_i under consideration miss the boundaries of both N_j and N_{j+1}. The fact that the components are interior to T' implies the boundaries of these components also miss all previous stages of (N_k).

Proof of Theorem 4.1. By assumption, i and j are chosen so that M_{i-1} is in N_1 and N_{j-1} is in M_1. Let n be a fixed integer. By Lemma 4.2, there is a homeomorphism h_1 of S^3 to itself, fixed on $X \cup (S^3 - M_1) \cup (S^3 - N_1)$, so that $h_1(\partial(M_i)) \cap \partial(N_j) = \emptyset$. This implies that each component S of $h_1(M_i)$ is either contained in the interior of a component of S' of N_j, or contains components of N_j.

Assume that S is contained in a component S' of N_j. By Lemma 4.3, there is a homeomorphism h_2 of S^3, fixed on X and the complement of S' so that $\partial(h_2(S))$ does not intersect $\partial(N_j) \cup \partial(N_{j+1})$. Either $h_2(S)$ is contained in a component S'' of N_{j+1} or it contains components of N_{j+1}.

Continue inductively applying Lemma 4.3 until a stage is reached so that the image of S under the composition of the homeomorphisms at each stage, $h(S)$, contains components $T'_1, \ldots, T'_r$ of some $N_{j+\ell}$, and so that $\partial(h(S))$ does not intersect $\partial(N_j) \cup \partial(N_{j+1}) \cup \ldots \cup \partial(N_{j+\ell})$. Such a stage must be reached because every time a Bing construction occurs in the defining sequence (N_k), components of (N_k) at that stage contain fewer components of the image of M_i than at the previous stage.

At this point, apply Lemma 4.4 to get a homeomorphism h' of S^3, fixed on X and on the complement of $h(S)$, so that $h' \circ h(\partial M_{i+1}) \cap \partial(\cup_{k=1}^r T'_k) = \emptyset$. We then have that the boundaries of $h' \circ h(S)$, and the boundaries of $h' \circ h$ of all components of M_{i+1} contained in X are disjoint from $\partial(N_j) \cup \partial(N_{j+1}) \cup \ldots \cup \partial(N_{j+\ell})$.

Do the above procedure for each component of $h_1(M_i)$ that is contained in a component of N_j. Do a similar procedure, starting with Lemma 4.4 for each component of $h_1(M_i)$ containing components of N_j. The result is a homeomorphism h_3 of S^3, fixed on X and on the complement of $h_1(M_i) \cup N_j$, so that $h_3 \circ h(\partial M_i \cup \partial M_{i+1}) \cap (\partial N_j \cup \partial N_{j+1}) = \emptyset$.

Next, repeat the entire above argument, starting with the fact that the boundaries of the image of M_{i+1} are disjoint from the boundaries of N_{j+1}. Continue inductively until a homeomorphism h of S^3 to itself, fixed on $X \cup (S^3 - M_1) \cup (S^3 - N_1)$ is obtained, so that $h(\partial(M_{i+m})) \cap \partial(N_{j+\ell}) = \emptyset$ for each nonnegative m and ℓ less than n. $\square$

5. Proof of the Main Result

As a special case, we first consider two Bing-Whitehead defining sequences for the same Bing-Whitehead compactum with the same initial stage.

Lemma 5.1. *Assume that X is a Bing-Whitehead compactum with two defining Bing-Whitehead sequences (M_i) and (N_j) and with $M_0 = N_0$. Then there is a homeomorphism of $M_0 = N_0$ that is fixed on $\partial M_0 = \partial N_0$ and on X that takes M_i onto N_i for any specified finite number of stages. In particular, if $X = BW(n_1, n_2, \ldots)$ with respect to (M_i), and $X = BW(m_1, m_2, \ldots)$ with respect to (N_j), then $m_i = n_i$ for all i.*

Proof. Suppose that such a homeomorphism h_n exists that matches the components up through n stages. Let T be a component of N_n. Let M equal $h_n(M_{n+1}) \cap T$ and N equal $N_{n+1} \cap T$. By Lemma 4.2 we may assume that the boundaries of M and N are disjoint.

It must be the case that M and N both have the same number of components. To see this, suppose M has one component and N has two. If M lies in a component of N then the geometric index of M in T would be 0 instead of 2. If M does not lie in a component of N, N must lie in the interior of M and by Theorem 3.5, M would be parallel to ∂T and its geometric index in T would be 1 instead of 2.

In case M and N both have one component, suppose that M lies in N, then ∂N is parallel to ∂T or ∂M. But the geometric index of N in T is 2 so ∂M and ∂N are parallel and the boundaries can be matched up with a homeomorphism of T taking ∂M to ∂N fixed on X and ∂T. The same argument works if N lies in M.

Suppose now that M and N both have two components. Then one component of M contains or is contained in one component of N and the other component of M contains or is contained in the other component of N. Theorem 3.5 can be used to show that ∂M and ∂N are parallel and as before we can get a homeomorphism fixed on X and ∂T taking M to N.

Repeating this argument in each component of N_n gives the homeomorphism h_{n+1}. $\qquad\square$

Note that the above proof also establishes the following Lemma.

Lemma 5.2. *Assume that X is a Bing-Whitehead compactum with two defining sequences (M_i) and (N_j) . If some component M of M_i is the same as some component N of N_j, then for all $k > 0$, $\alpha_{i+k} = \beta_{j+k}$*

where (α_ℓ) is the BW pattern for (M_i) and (β_ℓ) is the BW pattern for (N_j).

We next show that even without the same starting point, there is a component of some stage of one of the defining sequences that matches up with a component of the other defining sequence.

Lemma 5.3. *Assume that X is a Bing-Whitehead compactum with two defining Bing-Whitehead sequences (M_i) and (N_j). Let stages M_m and N_n be chosen so that they miss the compactum at infinity of the other stage. Suppose that T is a component of N_n in the interior of some component of M_m. Then there is a homeomorphism of S^3, fixed on X, taking T homeomorphically onto a component of some stage of $M_{m+\ell}$ for some $\ell \geq 0$.*

Proof. We choose a k so that $M_{m+k} \subset \operatorname{Int} N_n$. By Theorem4.1 we may assume that ∂T misses ∂M_i, $m \leq i \leq m + k$. Since $T \subset M_m$ and T does not lie in a component of M_{m+k}, we can find the largest subscript r so that T does lie in a component of M_r. Let S_0 be the component of M_r that contains T and S_1 be $M_{r+1} \cap S_0$. So S_1 is either a Bing link or a Whitehead link in S_0. If S_1 is a Whitehead link, then $S_1 \subset \operatorname{Int} T$ and by Theorem 3.4 ∂T is boundary parallel to either ∂S_0 or ∂S_1. In this case we may now assume by a homeomorphism fixing X that T equals S_0 or S_1. In case S_1 is a Bing link, then at least one and possibly both components of S_1 lie in $\operatorname{Int} T$. If both lie in $\operatorname{Int} T$, then ∂T and ∂S_0 are parallel by Theorem 3.5. If one component $S_1' \subset \operatorname{Int} T$ and the other component misses T, then ∂T and $\partial S_1'$ are parallel by Theorem 3.6. In either case, we may assume by a homeomorphism fixing X that T is either S_0 or S_1'. □

The previous lemmas can now be used to provide a proof of the main theorem.

Proof of Theorem 2.5

Let X_1 be a Bing-Whitehead Cantor set associated with a defining sequence (M_i) and let X_2 be a Bing-Whitehead Cantor set associated with a defining sequence (N_j). Assume that X_1 and X_2 are equivalently embedded. Then there is a homeomorphism of S^3 taking X_1 to X_2, so without loss of generality, we may assume $X = X_1 = X_2$ and that X has two Bing-Whitehead defining sequences (M_i) and (N_j). Let $(\alpha_1, \alpha_2, \alpha_3, \ldots)$ be the BW pattern of X with respect to (M_i) and let $(\beta_1, \beta_2, \beta_3, \ldots)$ be the BW pattern of X with respect to (N_j). Choose stages M_m of (M_i) and N_n of (N_j) so that

- M_m is contained in N_1 and N_n is contained in M_1.

- Both M_m and N_n have 2^r components, and both M_{m+1} and N_{n+1} are obtained by placing Bing constructions in each component of the previous stage.

Apply Lemma 4.2 to adjust M_m and N_n so that their boundaries do not intersect. If all the components of M_m are contained in components of N_n, then the components must match up in a 1-1 fashion, and the proof of Lemma 5.3, together with the fact that the next stage is a Bing construction, shows that there is a homeomorphism matching up these components. Then by Lemma 5.2, $\beta_{n+k} = \alpha_{m+k}$ for all $k \geq 0$, establishing the needed result. A similar argument gives this conclusion if all the components of N_n are contained in components of M_m.

If some component of M_m contains more than one component of N_n, then some component of N_n also contains more than one component of M_m. Let T_1 be a component of N_n contained in some component of M_m. By Lemma 5.3, T_1 can be matched homeomorphically with a component of some M_{m+p} and so by Lemma 5.2, $\beta_{n+k} = \alpha_{m+p+k}$ for all $k \geq 0$. Let T_2 be a component of M_m contained in some component of N_n. By Lemma 5.3, T_2 can be matched homeomorphically with a component of some N_{n+q} and so by Lemma 5.2, $\alpha_{m+k} = \beta_{n+q+k}$ for all $k \geq 0$. Thus

$$\alpha_{m+k} = \beta_{n+q+k} = \alpha_{m+p+q+k} = \alpha_{(m+k)+(p+q)}$$

If $p > 0$ or $q > 0$, this implies the BW pattern for X with respect to (M_i) is repeating, contradicting the fact that $\sum_i n_i 2^{-i}$ diverges where $X = BW(n_1, n_2, \ldots)$ with respect to (M_i). Thus $p = q = 0$ and $\beta_{n+k} = \alpha_{m+k}$ for all $k \geq 0$, establishing the needed result.

□

6. QUESTIONS

(1) Is it possible to generalize the main theorem (Theorem 2.5) to apply to the construction of DeGryse and Osborne in dimensions greater than three?

(2) Is it possible to distinguish Bing-Whitehead compacta that vary the placement of Bing and Whitehead constructions at each stage, rather than using all Bing or all Whitehead constructions at each stage?

(3) Is it possible to use the techniques of the main theorem to construct rigid Cantor sets of genus one in S^3 with simply connected complements? See [GRŽ06] for a discussion of rigid Cantor Sets.

7. Acknowledgments

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