# ANALYSIS OF AN INTEGRO–DIFFERENTIAL EQUATION ARISING FROM MODELLING OF FLOWS WITH FADING MEMORY THROUGH FISSURED MEDIA

MAŁGORZATA PESZYŃSKA

Abstract. An analysis of an integro–differential equation with a convolution term is given. Such equations arise in modelling of flows through fissured media, and these integral terms account for fading memory effects exhibited by the flow. We propose a convergent semi–discrete approximation of the convolution term with a possibly singular kernel. The approximation scheme leads to the existence/uniqueness result for the problem and has strongly favorable numerical aspects.

## 1. Introduction

Models of microstructure phenomena have recently attracted much interest. This is related to the appearance of new techniques of modelling like homogenization methods (see [5, 17]) and new achievements in numerical methods, especially recent developments in parallel computing. The models of flows through fissured media are examples of microstructure models which require nonstandard techniques for their analysis and approximation.

Fissured media are porous media of hierarchical geometrical structure. Below we are concerned with a model of flow through fissured medium proposed and analysed by Hornung and Showalter in [9], see also [3], which is derived by homogenization and takes into account the fading memory effects exhibited by the flow. The equation we deal with has the form

$$u_t(x,t) + \int_0^t u_t(x,s)\tau(t-s)ds - \nabla \cdot (D\nabla u(x,t)) \quad = \quad f(x,t),\ (x,t) \in \Omega \times I \qquad (1)$$

$$u(x,t) \quad = \quad 0,\ (x,t) \in \partial\Omega \times I \qquad (2)$$

$$u(x,0) \quad = \quad u_0(x),\ x \in \Omega \qquad (3)$$

where an open bounded set $\Omega \subset R^d$, $d = 2$ or $d = 3$, with boundary $\partial\Omega$ is the domain of the flow. $I = (0,T), T > 0$ is the time interval, and D is the coefficient tensor (possibly dependent on space variable). The convolution kernel $\tau(\cdot)$ is related to the microscopic properties of the domain of the flow and is by definition (see [9, 15]) singular at $t = 0$ but $L^1$ integrable. For example, for some particular microscopic geometric structure of the medium the convolution kernel is given by

$$\tau(t) \quad = \quad c_0 \sum_{k=1}^{\infty} e^{-c_1 k^2 t}$$

1991 *Mathematics Subject Classification.* Primary: 35K99, Secondary: 45K05,35A35.

*Key words and phrases.* integro–partial differential equations, existence and uniqueness of solutions, convolution integrals.

where the coefficients $c_0, c_1$ depend on the properties of the medium (see [15]).

We note that in the case $\tau \equiv 0$ the equation ( 1 ) reduces to the standard linear diffusion equation. The same reduction is valid if $\tau$ is given as the Dirac measure concentrated at $t = 0$. Both of these trivial cases describe diffusion with the postulated instantaneous propagation of changes in the values of variables. The nature of the flow in fissured media requires $\tau \not\equiv 0, \tau \not\equiv \delta$ i.e. the presence of nontrivial memory terms in the model. In the case considered in this paper those memory terms admit the convolution representation with kernels unbounded at the origin. It has been observed that the models with memory terms with bounded kernels lead to a less accurate description of the dynamics of the flow than is the case treated in this paper, i.e., the case of singular kernels (see [9, 8]). We mention also the work of Arbogast and Douglas in [1, 2] who use a different modelling approach in order to describe the memory effects. Their work however can be imbedded in a common framework together with the model above by means of an application of a generalized convolution operator proposed in [15].

The models corresponding to the case when ( 1 ) reduces to the linear diffusion equation yield to the standard analytic and numerical treatment. In this paper we are focussed on the proper treatment of the memory terms with nontrivial kernels. We propose a convergent approximation scheme which combined with the method of lines (also known as Rothe method) leads to the existence/uniqueness result for the model. The advantage of our method with respect to the one applied in [9] is in the less restrictive assumptions on the elliptic part of the problem. Also, our approximation algorithm has a strong numerical aspect as it provides a tool for numerical treatment of integral terms with singular kernels (see [15, 16] in this direction). Such integral terms appear frequently in viscoelasticity theory ([13] and references given there), theory of phenomena with memory ([14]) as well as in homogenization limits of scalar conservation laws ([18]). The method presented below has many potential analytical and numerical applications.

The paper is organized as follows. In Section 2 we analyse properties of the convolution term under relatively weak assumptions on the convolution kernel. In Section 3 we define a method of approximation of the convolution term and formulate basic technical lemmas. In Section 4 we prove the main result of this paper on existence of a unique weak solution to the problem ( 1 )–( 3 ).

## 2. Convolution operator

In this section we recall and study properties of the convolution operator $L_\tau : L^2(I) \mapsto L^2(I)$ or, in general $L_\tau : L^2(I;\mathcal{H}) \mapsto \mathcal{L}^{\in}(\mathcal{I};\mathcal{H})$ with $\mathcal{H}$ a Hilbert space with scalar product $(\cdot,\cdot)_\mathcal{H}$ and norm $\|\cdot\|_\mathcal{H})$. Let $\tau \in L^1(I)$ be fixed. The operator $L_\tau$ is defined by

$$(L_\tau v)(t) = (\tau \star v)(t) = \int_0^t \tau(t-s)v(s)ds \quad , \quad v \in L^2(I), t \in I.$$

Recall that the convolution product is symmetric and that formally

$$\frac{d}{dt}(f_1 \star f_2)(t) = f_1(0)f_2(t) + (\frac{df_1}{dt} \star f_2)(t) = f_2(0)f_1(t) + (\frac{df_2}{dt} \star f_1)(t)$$

$$(f_1 \star \frac{df_2}{dt})(t) = f_1(0)f_2(t) - f_2(0)f_1(t) + (\frac{df_1}{dt} \star f_2)(t).$$

Since (see for example [7])

$$\| L_\tau v \|_{L^2(I)} = \| \tau \star v \|_{L^2(I)} \le \| \tau \|_{L^1(I)} \| v \|_{L^2(I)},$$

it follows that if $\tau \in L^1(I)$ then $L_\tau$ is a linear continuous operator. In our analysis we shall use some nontrivial properties of the convolution operator, precisely its monotonicity and another property which we call the $\mathcal{LTD}$ property.

Monotonicity of the convolution operator means that for an arbitrary $v \in L^2(I;\mathcal{H})$ we have

$$\int_0^t ((L_\tau v)(s), v(s))_{\mathcal{H}}\, ds \ge 0 \quad , \quad \forall t \in I,$$

while the $\mathcal{LTD}$ property is the feature of convolution operators expressed by the inequality

$$\int_0^t ((L_\tau v_t)(s), v(s))_{\mathcal{H}}\, ds \ge -C_{\mathcal{LTD}} \| v(0) \|^2_{\mathcal{H}},$$

where $C_{LTD}$ is a positive constant independent of $v$. The notion $\mathcal{LTD}$ (an abbreviation for *like time derivative*) is related to the analogy with the following inequality easily derived for the identity operator in the place of $L_\tau$

$$\int_0^t (v_t(s), v(s))_{\mathcal{H}}\, ds = \frac{1}{2} \| v^2(t) \|_{\mathcal{H}} - \frac{1}{2} \| v^2(0) \|_{\mathcal{H}} \ge -\frac{1}{2} \| v^2(0) \|_{\mathcal{H}} .$$

The following assumption on the convolution kernel is sufficient to yield both mentioned features.

**Assumption 2.1.** *$\tau \in L^1(I) \cap C^1(R^+)$, $\tau$ is a nonnegative nonconstant function with a nonpositive nondecreasing derivative $\tau'$.*

In [12] MacCamy et al proved that Assumption ( 2.1 ) guarantees the *(strong) positivity* of the convolution kernel and in consequence monotonicity of the convolution operator. Below we prove that this Assumption implies the $\mathcal{LTD}$ property. Our proof follows the idea outlined in the monograph [7], where also more general properties of monotone kernels are considered.

**Lemma 2.2.** *Let $\tau$ satisfy Assumption ( 2.1 ). Then for $v \in H^1(I)$ there holds*

$$(L_\tau v_t, v)_{L^2(I)} \ge -\frac{\| \tau \|_{L^1(I)}}{2} v^2(0). \tag{4}$$

*Proof.* Define $\gamma$ to be the (Borel) measure induced by the (distributional) derivative $\tau'$ of $\tau$ i.e. $\gamma([0,t]) = \tau(t)$. Recall that this formally implies, with $\delta$ denoting Dirac measure, that

$$\int_{[0,t]} f(s)\gamma(ds) = \int_0^t f(s)\tau'(s)ds + \tau(0)f(t).$$

In order to prove ( 4 ) we shall use the identity

$$\begin{aligned}\mathcal{V} &\stackrel{\text{def}}{=} (L_\tau v_t, v)_{L^2(I)} \\ &= \int_0^T \tau(T-t)\frac{v^2(t)}{2}dt - \int_0^T \tau(T-t)\frac{v^2(0)}{2}dt + \int_0^T \tau(t)\frac{(v(0)-v(t))^2}{2}dt \\ &\qquad - \int_{[0,T]}\int_s^T \frac{(v(t-s)-v(t))^2}{2}dt\gamma(ds).\end{aligned}$$

Except for the second term, all terms on the right hand side of this identity are positive (note that since $\tau$ is monotone decreasing, $(-\gamma)$ is a positive measure and as $s = 0$, the integrand $v(t-s) - v(t) = 0$). Hence

$$\mathcal{V} \geq -\int_0^T \tau(T-t)\frac{v^2(0)}{2}dt = \frac{v^2(0)}{2}\int_0^T \tau(T-t)dt = -\frac{v^2(0)}{2} \parallel \tau \parallel_{L^1(I)}$$

and the Lemma is proved. It remains then to prove ( 5 ). Rewriting the r.h.s. and l.h.s of this identity we get, respectively

$$\left(\int_0^T \tau(T-t)\frac{v^2(t)}{2}dt - \int_0^T \tau(T-t)\frac{v^2(0)}{2}dt\right)$$
$$+\left(\int_0^T \tau(t)\frac{v^2(0)}{2}dt + \int_0^T \tau(t)\frac{v^2(t)}{2}dt - \int_0^T v(0)v(t)\tau(t)dt\right)$$
$$-\left(\int_0^T \frac{v^2(s)}{2}\tau(T-s)ds + \int_0^T \frac{v^2(s)}{2}\tau(s)ds - \int_{[0,T]}\int_s^T v(t-s)v(t)dt\gamma(ds)\right)$$

$$\begin{aligned}\mathcal{V} = &\int_0^T \tau(0)v^2(t)dt - \int_0^T \tau(t)v(t)v(0)dt \\ &+ \int_0^T \int_{[0,s]} v(s-t)v(s)\gamma(dt)ds - \int_0^T v^2(s)\tau(0)ds \\ &= -\int_0^T \tau(t)v(t)v(0)dt + \int_0^T \int_{[0,s]} v(s-t)v(s)\gamma(dt)ds.\end{aligned}$$

After reduction of the terms which cancel pairwise on both sides it remains to prove that the first term on the r.h.s. equals the last term on the l.h.s. i.e.

$$\int_{[0,T]}\int_s^T v(t-s)v(t)dt\gamma(ds) = \int_0^T \int_{[0,s]} v(s-t)v(s)\gamma(dt)ds.$$

This last identity is a consequence of the change of order of integration of type

$$\int_0^T \int_s^T f(t,s)dtds = \int_0^T \int_0^s f(s,t)dtds.$$

We note that the Lemma is valid for every $\tilde{I} = (0, \tilde{T}) \subset I$, with $\tilde{T} < T$. □

**Corollary 2.3.** *Let $\tau$ satisfy Assumption ( 2.1 ) and let $u \in H^1(I; L^2(\Omega))$. Then for every $t \in I$ holds*

$$\int_0^t ((L_\tau u_t(s), u(s))_{L^2(\Omega)}\, ds \geq -\frac{\mathcal{T}(t)}{2} \parallel u(0) \parallel^2_{L^2(\Omega)} \tag{6}$$

*where $\mathcal{T}(t) \stackrel{\text{def}}{=} \int_0^t \tau(s)ds = \parallel \tau \parallel_{L^1(0,t)}$.*

*Proof.* Using Lemma ( 2.2 ) we can write, for fixed $x \in \Omega$ treated as a parameter

$$\int_0^t (L_\tau \frac{du}{dt})(s,x)u(s,x)ds \geq -\frac{\mathcal{T}(t)}{2}|u(0,x)|^2 \quad , \ \ \forall x \in \Omega, u \in H^1(I; L^2(\Omega)). \tag{7}$$

Integrating ( 7 ) over $\Omega$ yields the desired result ( 6 ). □

There exists an interesting alternative proof of the $\mathcal{LTD}$ property for convolution kernels arising in microstructure models of flows through fissured media (see [15]). The proof appeals to some relations induced directly by the nature of the described phenomenon and implicitly implies $C_\tau \leq 1$.

## 3. Approximation of convolution term

In this section we define the discrete approximation of the convolution term which is a basic tool in the proof of our main result in Section 4. We also give some technical results useful in the sequel.

We define the partition of $I$ as follows: let $n > 0$ be integer, $h = \frac{T}{n}$ be the *time step* (called also *time discretization parameter*), let $t_k = kh, k = 0, \ldots n$, and

$$I = \bigcup_{k=1}^{n} I_k \quad , \quad I_k = (t_{k-1}, t_k].$$

For an integrable function $g$ its piecewise constant approximation $\bar{g}_n(s)$ in $I$ is defined as $g_k$ whenever $s \in I_k, 1 \leq k \leq n$ with

$$g_k = \frac{1}{h} \int_{t_{k-1}}^{t_k} g(s)ds \quad , \quad 1 \leq k \leq n.$$

Let $\tau$ be fixed and satisfy Assumption ( 2.1 ) and let $L_\tau$ be the convolution operator with kernel $\tau$. We define the discrete counterpart $\bar{L}_n$ of $L_\tau$ as

$$\left(\bar{L}_n v\right)(s) = (L_k v)(s) \text{ whenever } s \in I_k \quad , \quad v \in L^2(I),$$

where

$$L_k v = (L_\tau v)_k = \frac{1}{h} \int_{t_{k-1}}^{t_k} (L_\tau v)(s)ds \quad , \quad v \in L^2(I).$$

Denote by $\theta_k$ the characteristic function of the subinterval $I_k$ and define a family of functions $\{\xi_k\}_{k=1}^n$ as follows:

$$\xi_k(s) = \int_0^s \tau(s-r)\theta_k(r)dr.$$

Directly from the definition of $(\xi_k)_{k=1}^n$ we obtain

$$\xi_{k+1}(t) = \xi_k(t-h) \quad , \quad t \geq t_{k+1} \quad , \quad 0 \leq k \leq n-1. \tag{8}$$

Moreover, all elements of the family $\{\xi_k\}_{k=1}^n$ are nonnegative and since $\sup_{t \in I} \xi_k(t) = \xi_k(t_k)$ they are uniformly bounded by the value $c_\tau = \mathcal{T}(T)$. We define also

$$\eta_{i,k} = \frac{1}{h} \int_{t_{i-1}}^{t_i} \xi_k(t)dt \quad , \quad 1 \leq i, k \leq n.$$

Using ( 8 ) we immediately prove that the coefficients $\eta_{i,k}, i, k = 1, \ldots n$ are non-negative and uniformly bounded by $c_\tau$. They form a Toeplitz matrix and for every

$1 \le i,k \le n$ the following properties hold:

$$\eta_{i,k} = 0 \quad , \quad i < k, \tag{9}$$
$$\eta_{1,1} = \eta_{2,2} = \dots \eta_{n,n} > 0 \tag{10}$$
$$\sum_{k=1}^{n} \eta_{i,k} \le c_\tau \tag{11}$$
$$\eta_{i,k} = \eta_{i-1,k-1} = \dots \eta_{i-k+1,1} \quad , \quad i \ge k, \tag{12}$$
$$\eta_{i,1} < \eta_{i,2} < \dots \eta_{i,i}. \tag{13}$$

Now we set $\vartheta_{j,k} \stackrel{\text{def}}{=} \eta_{j-1,k} - \eta_{j,k}$ and from the above properties we conclude that $\vartheta_{j,k} \ge 0$. Also

$$\gamma_j = \sum_{k=1}^{j-1} \vartheta_{j,k} = \sum_{k=1}^{j-1} (\eta_{j-1,k} - \eta_{j,k}) \le \eta_{j,j} \quad , \quad 2 \le j \le n. \tag{14}$$

The latter bound follows from

$$\begin{aligned} \sum_{k=1}^{j-1} \eta_{j-1,k} &= \sum_{k=1}^{j-1} \frac{1}{h} \int_{t_{j-2}}^{t_{j-1}} \xi_k(t)dt = \frac{1}{h} \int_{t_{j-2}}^{t_{j-1}} \sum_{k=1}^{j-1} \int_0^t \tau(t-s)\theta_k(s)dsdt \\ &= \frac{1}{h} \int_{t_{j-2}}^{t_{j-1}} \int_0^t \tau(s)dsdt \end{aligned}$$

and

$$\sum_{k=1}^{j-1} \eta_{j,k} = \frac{1}{h} \int_{t_{j-1}}^{t_j} \int_0^t \tau(t-s)\chi_{[0,t_{j-1}]}(s)dsdt = \frac{1}{h} \int_{t_{j-1}}^{t_j} \int_{t-t_{j-1}}^t \tau(s)dsdt$$

which yield

$$\gamma_j = \sum_{k=1}^{j-1} \eta_{j-1,k} - \eta_{j,k} = \frac{1}{h} \int_{t_{j-2}}^{t_{j-1}} \int_0^t \tau(s)dsdt - \frac{1}{h} \int_{t_{j-1}}^{t_j} \int_{t-t_{j-1}}^t \tau(s)dsdt.$$

From the definition of $\eta_{j,j}$ we get

$$\gamma_j = \eta_{j,j} + \frac{1}{h} \int_{t_{j-2}}^{t_{j-1}} \int_0^t \tau(s)dsdt - \frac{1}{h} \int_{t_{j-1}}^{t_j} \int_0^t \tau(s)dsdt.$$

To prove ( 14 ) it suffices now to show

$$\frac{1}{h} \int_{t_{j-2}}^{t_{j-1}} \int_0^t \tau(s)dsdt \le \frac{1}{h} \int_{t_{j-1}}^{t_j} \int_0^t \tau(s)dsdt, \tag{15}$$

a consequence of the fact that the function $\mathcal{T}(\cdot)$ as integral of a nonnegative function is continuous, positive and increasing . Hence ( 15 ) holds and ( 14 ) follows.

The convolution integral and its discrete counterpart take an especially convenient form for step functions, i.e., if $v(t) = \sum_{k=1}^n v_k \theta_k(t)$ for $t \in I$. In this case we have

$$(L_\tau v)(s) = \sum_{k=1}^{n} v_k \xi_k(s) \quad , \quad \forall s \in I,$$

and

$$(\bar{L}_n v)(s) = L_i v = \sum_{k=1}^{i} \eta_{i,k} v_k \quad \text{whenever } t_{i-1} < s \leq t_i.$$

In the sequel the above representation shall be used frequently. We shall use also the technical result below which shows the convergence of approximation of integrable functions by step functions.

**Lemma 3.1.** *Let $g \in L^2(I; L^2(\Omega))$ and let $\bar{g}_n$ be its piecewise constant approximation in $I$. Then $\bar{g}_n \to g$ (strongly) in $L^2(I; L^2(\Omega))$, i.e.,*

$$\int_0^T \| \bar{g}_n(s, \cdot) - g(s, \cdot) \|^2_{L^2(\Omega)} \, ds \to 0. \tag{16}$$

*Moreover, if $g \in H^1(I; L^2(\Omega))$, then there exists a constant $c$ dependent on $\| \frac{dg}{dt} \|_{L^2(I;L^2(\Omega))}$ such that*

$$\int_0^T \| \bar{g}_n(s, \cdot) - g(s, \cdot) \|^2_{L^2(\Omega)} \, ds \leq ch^2. \tag{17}$$

*Proof.* The result ( 16 ) is standard (see e.g. [11, 6, 15]). The second part follows from the definition of $\bar{g}_n$ and the estimates

$$\| g(s, \cdot) - \bar{g}_n(s, \cdot) \|^2_{L^2(\Omega)} \leq \frac{1}{h} \int_{I_k} \| g(s, \cdot) - g(t, \cdot) \|^2_{L^2(\Omega)} \, dt \quad , \quad s \in I_k, 1 \leq k \leq n. \tag{18}$$

For $g \in H^1(I; L^2(\Omega))$ we have

$$g(s, x) - g(t, x) = \int_t^s \frac{dg}{dt}(r, x) dr \quad , \quad \text{a.e. } x \in \Omega.$$

Estimating ( 18 ) we obtain, for $s \in I_k$

$$\| g(s, \cdot) - \bar{g}_n(s, \cdot) \|^2_{L^2(\Omega)} \leq \frac{1}{h} \int_{I_k} \| \int_t^s (\frac{dg}{dt})(r, \cdot) dr \|^2_{L^2(\Omega)} \, dt$$
$$\leq h \int_{I_k} \| (\frac{dg}{dt})(r, \cdot) \|^2_{L^2(\Omega)} \, dr$$

and finally

$$\int_0^T \| g(s, \cdot) - \bar{g}_n(s, \cdot) \|^2_{L^2(\Omega)} \, ds \leq h \sum_k \int_{I_k} \| (\frac{dg}{dt})(r, \cdot) \|^2_{L^2(\Omega)} \, dr$$
$$\leq h^2 \sum_k \int_{I_k} \| (\frac{dg}{dt})(r, \cdot) \|^2_{L^2(\Omega)} \, dr = h^2 \int_0^T \| (\frac{dg}{dt})(r, \cdot) \|^2_{L^2(\Omega)} \, dr,$$

hence, we have obtained ( 17 ). □

## 4. Existence/uniqueness result

In this section we prove existence of a unique (weak) solution to the problem ( 1 )–( 3 ). We set

$$H = L^2(\Omega), V = H_0^1(\Omega)$$

and denote by $(\cdot, \cdot)$ the scalar product in $H$ (identified with the duality pairing between $H$ and $H^*$) and by $|\cdot|$, $\|\cdot\|$ the norms in $H, V$, respectively. Note that by the assumptions on $\Omega$ as in the Introduction the imbedding $V \subset H$ is dense and compact.

We shall be concerned with a solution of the problem

$$(u_t, v) + (L_\tau u_t, v) + a(u, v) = (f, v) \quad , \quad \forall v \in V, \forall t \in I \tag{19}$$

$$(u(0), v) = (u_0, v) \quad , \quad \forall v \in V \tag{20}$$

which is the weak form of ( 1 )–( 3 ) with bilinear form $a(\cdot, \cdot)$ defined by

$$a(u, v) = \sum_{i,j=1}^{d} D_{i,j} \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}(x) dx \quad , \quad u, v \in V.$$

The proof of existence of a (unique) solution to this problem consists of several steps, which, up to the special treatment of the convolution term, follow the general structure of proofs known as Rothe method (method of lines) or method of fractional steps (e.g. [10, 11]). At first, using the discrete counterpart of the convolution term defined in Section 3, we construct a sequence approximating the solution of the problem ( 19 )–( 20 ), whose elements are defined as solutions of appropriately chosen boundary value problems. Next we prove the convergence of this sequence by using some a-priori estimates and the technique similar to Ascoli–Arzelá lemma. In the end we show that our approximation scheme is consistent i.e. that the limit of the sequence of approximations satisfies the problem ( 19 )–( 20 ).

Below we restrict ourselves to the case where the following assumptions are satisfied:

- the convolution kernel $\tau$ satisfies Assumption ( 2.1 ),
- tensor $\mathbf{D}$ is constant, symmetric and satisfies the ellipticity condition

$$\sum D_{ij} \xi_i \xi_j \geq c_D \sum |\xi_i|^2 \quad , \quad \forall \xi = (\xi_i)_{i=1}^d \quad , \quad \xi \neq 0,$$

  with a positive constant $c_D$,
- $f \in H^1(I; H), f(0) \in H$,
- $u_0 \in V$ and we have $\sup_{|v| \leq 1, v \in H} a(u_0, v) < \infty$.

We note that under above assumptions the convolution operator $L_\tau$ is continuous, monotone and satisfies the $\mathcal{LTD}$ property, while $a(\cdot, \cdot)$ is symmetric, continuous and elliptic, i.e., there exist constants $c_a, k_a > 0$ such that

$$a(u, u) \geq c_a \parallel u \parallel^2 \quad , \quad \forall u \in V \tag{21}$$

$$a(u, v) \leq k_a \parallel u \parallel \parallel v \parallel \quad , \quad \forall u, v \in V. \tag{22}$$

We remark that the assumptions on $D$ (made above for simplicity), are not crucial for our results, as it can be seen from the proof below. They could be relaxed to more general form of $a(\cdot, \cdot)$ (in particular, nonsymmetric form with space variable dependent coefficients).

**Lemma 4.1.** *Problem ( 19 )-( 20 ) has at most one solution.*

*Proof.* Assume that $u_1, u_2$ are two solutions of ( 19 )-( 20 ). Then, subtracting equation ( 19 ) rewritten for $u_1, u_2$, respectively, we get

$$((u_1 - u_2)_t, v) + (L_\tau (u_1 - u_2)_t, v) + a(u_1 - u_2, v) = 0 \quad , \quad \forall v \in V.$$

Take $v = u_1 - u_2$ and integrate over $(0,t)$ to obtain

$$\frac{1}{2}|u_1-u_2|^2(t) - \frac{1}{2}|u_1-u_2|^2(0) + \int_0^t (L_\tau(u_1-u_2)_t, u_1-u_2)(s)ds + \int_0^t a(u_1-u_2,u_1-u_2)(s)ds = 0. \tag{23}$$

By the $\mathcal{LTD}$ property of operator $L_\tau$, initial condition $u_1(0) = u_2(0)$, ellipticity of $a(\cdot,\cdot)$ we infer

$$\frac{1}{2}|u_1-u_2|^2(t) + c_a \int_0^t \| u_1 - u_2 \|^2 (s)ds \leq 0.$$

Hence $u_1(t) = u_2(t), \forall t \in I$. □

Below we define the approximation scheme.

**Definition 4.2.** *Define* $u_i \in V, 1 \leq i \leq n$ *as the solutions of*

$$(\partial_h u_i, v) + \sum_{k=1}^{i} \eta_{i,k}(\partial_h u_k, v) + a(u_i, v) = (f_i, v) \quad , \quad v \in V, 1 \leq i \leq n \tag{24}$$

*with*

$$\partial_h u_i \overset{\text{def}}{=} \frac{1}{h}(u_i - u_{i-1}), \quad , \quad 1 \leq i \leq n,$$

*where* $u_0$ *is identified with the initial value specified in ( 20 ) and* $f_i$ *is the piecewise constant approximation of* $f$.

Let us remark that the sequence $u_i, 1 \leq i \leq n$ is well defined, i.e., for each $1 \leq i \leq n$ there exists a unique solution $u_i \in V$ of the equation ( 24 ). To see this we define the operator $A_i^h : V \to V^*$ by

$$(A_i^h u, v) = \frac{1+\eta_{i,i}}{h}(u,v) + a(u,v)$$

which by ( 21 )–( 22 ) is continuous and $V$-coercive. Letting

$$F_i = f_i - \sum_{k=1}^{i-1} \eta_{i,k} \partial_h u_k + \frac{1+\eta_{i,i}}{h} u_{i-1}$$

with $F_i \in H \subset V^*$, from the Lax-Milgram theorem we infer the existence of a unique solution of the problem

$$(A_i^h u_i, v) = (F_i, v) \quad , \quad \forall v \in V,$$

an equivalent of ( 24 ).

Below we prove boundedness of the sequence $(u_i)_{i=1}^n$.

**Lemma 4.3.** *Let the sequence* $(u_i)_{i=1}^n$ *be defined by ( 24 ). Then for* $2 \leq j \leq n$

$$| \partial_h u_j | \leq h \, | \partial_h f_j | + \max_{k=1\ldots j-1} | \partial_h u_k | \tag{25}$$

*and for* $j = 1$

$$| \partial_h u_1 | \leq | f(0) | + h \, | \partial_h f_1 | + c_{a,u_0} \tag{26}$$

*with* $c_{a,u_0} = \sup_{|v| \leq 1, v \in H} a(u_0, v) < \infty$ *and* $\partial_h f_1 = \frac{f_1 - f(0)}{h}$.

*Proof.* Let $j \geq 2$. We rewrite ( 24 ) for $i = j, i = j-1$, subtract the two equations and get

$$\begin{aligned}(1 \quad + \quad & \eta_{j,j})(\partial_h u_j, v) + a(u_j - u_{j-1}, v) \\ = \quad & (f_j - f_{j-1}, v) + (\partial_h u_{j-1}, v) - \sum_{k=1}^{j-1} \eta_{j,k}(\partial_h u_k, v) + \sum_{k=1}^{j-1} \eta_{j-1,k}(\partial_h u_k, v).\end{aligned}$$

Now we take $v = \partial_h u_j$, use the definition and properties of $\vartheta_{j,k}$ ( 14 ), ellipticity of the form $a(\cdot,\cdot)$, Cauchy-Schwarz inequality, Poincaré–Friedrichs inequality with the appropriate constant for simplicity set to identity, to get the estimate

$$\begin{aligned}(1 \quad + \quad & \eta_{j,j}) \mid \partial_h u_j \mid^2 + hc_a \mid \partial_h u_j \mid^2 \\ \leq \quad & h \mid \partial_h f_j \mid\mid \partial_h u_j \mid + \mid \partial_h u_{j-1} \mid\mid \partial_h u_j \mid + \mid \partial_h u_j \mid \sum_{k=1}^{j-1} \vartheta_{j,k} \mid \partial_h u_k \mid .\end{aligned} \tag{27}$$

If $\mid \partial_h u_j \mid = 0$ hypothesis of the Lemma follows. Otherwise we divide both sides of ( 27 ) by $\mid \partial_h u_j \mid$, use definition of $\gamma_j$ (see ( 14 )) and estimate

$$\sum_{k=1}^{j-1} \vartheta_{j,k} \mid \partial_h u_k \mid \leq \max_{k=1,\dots j-1} \mid \partial_h u_k \mid \sum_{k=1}^{j-1} \vartheta_{j,k}$$

to get

$$(1 + \eta_{j,j} + hc_a) \mid \partial_h u_j \mid \leq h \mid \partial_h f_j \mid + (1 + \gamma_j) \max_{k=1,\dots j-1} \mid \partial_h u_k \mid . \tag{28}$$

Since $c_a > 0, \eta_{j,j} \geq 0$ we have $\frac{1}{1+\eta_{j,j}+hc_a} < 1$, moreover, by ( 14 ) follows $\frac{1+\gamma_j}{1+\eta_{j,j}+hc_a} \leq 1$. Eliminating constants in ( 28 ) yields

$$\mid \partial_h u_j \mid \leq h \mid \partial_h f_j \mid + \max_{k=1,\dots j-1} \mid \partial_h u_k \mid,$$

and we have proved ( 25 ). The proof of ( 26 ) proceeds similarly. □

The following result is elementary.

**Lemma 4.4.** *Let $n > 0$ be integer and let us be given two sequences (finite) of positive numbers $(a_k)_{k=1}^n$, $(b_k)_{k=1}^n$ such that*

$$a_k \leq b_k + \max_{i=1,\dots k-1} a_i \quad , \quad \forall k = 1 \dots n.$$

*Then for each $1 \leq k \leq n$, $a_k \leq \sum_{i=1}^k b_i$.*

This result is applied to the estimates given in Lemma ( 4.3 ) with

$$\begin{aligned} a_i \quad &= \quad \mid \partial_h u_i \mid \quad , \quad i = 1, \dots n \\ b_i \quad &= \quad h \mid \partial_h f_i \mid \quad , \quad i = 2, \dots n \\ b_1 \quad &= \quad \mid f(0) \mid + h \mid \partial_h f_1 \mid + c_{a,u_0}\end{aligned}$$

with constant $c_{\partial u}$ set as

$$c_{\partial u} = c_{a,u_0} + \mid f(0) \mid + \sqrt{T} \parallel \frac{df}{dt} \parallel_{L^2(I;H)}$$

to obtain the following.

**Corollary 4.5.** *There exists a positive constant $c_{\partial u}$ independent of $n$ such that $\forall i = 1, \dots n$*

$$| \partial_h u_i | \leq c_{\partial u}. \tag{29}$$

Above we derived bounds in the norm of $H$; below we prove the bounds in $V$.

**Lemma 4.6.** *There exists a positive constant $c_u$ independent of $n$ such that $\forall j = 1, \dots n$ holds*

$$\| u_j \| \leq c_u. \tag{30}$$

*Proof.* To prove this estimate, we rewrite ( 24 ) for $i = j, v = u_j$ in the form

$$a(u_j, u_j) = (f_j, u_j) - (1 + \eta_{j,j})(\partial_h u_j, u_j) - \sum_{k=1}^{j-1} \eta_{j,k}(\partial_h u_k, u_j) \tag{31}$$

From ( 21 ), ellipticity of $a(\cdot,\cdot)$ and Cauchy-Schwarz inequality we estimate

$$c_a \| u_j \|^2 \leq | f_j || u_j | + (1 + \eta_{j,j}) | \partial_h u_j || u_j | + | u_j | \max_{k=1,\dots j-1} | \partial_h u_k | \sum_{k=1}^{j-1} \eta_{j,k} \tag{32}$$

Now,

$$f_j = f(0) + \sum_{k=1}^{j} \partial_h f_k$$

$$| f_j | \leq | f(0) | + \sqrt{T} \| \frac{df}{dt} \|_{L^2(I;H)} = c_f$$

hence, by ( 11 ) and ( 32 ) we infer

$$c_a \| u_j \|^2 \leq \| u_j \| \left[ c_f + (1 + 2c_\tau) c_{\partial u} \right].$$

The conclusion of the Lemma, i.e., the estimate ( 30 ), follows with

$$c_u = \frac{1}{c_a}(c_f + (1 + 2c_\tau)c_{\partial u}).$$

and this concludes the proof. □

Given $(u_i)_{i=0}^n$ we define the Rothe's function $u_n(t)$ and the corresponding step function $\bar{u}_n(t)$

$$u_n(t) = u_{i-1} + (u_i - u_{i-1}) \frac{t - t_{t-1}}{h} \quad , \quad t_{i-1} < t \leq t_i \quad , \quad i = 1 \dots n \tag{33}$$

$$\bar{u}_n(t) = u_i \quad , \quad t \in I_i \quad , \quad i = 1 \dots n. \tag{34}$$

As a consequence of Corollary ( 4.5 ) and Lemma ( 4.6 ) we have

**Corollary 4.7.** *Let $u_n, \bar{u}_n$ be defined by ( 33 ), ( 34 ) and constants $c_u, c_{\partial u}$ as in ( 29 ), ( 30 ). Then*

$$| \frac{du_n}{dt}(t) | \leq c_{\partial u} \quad , \quad a.e. \ t \in I,$$
$$\| \bar{u}_n(t) \| \leq c_u \quad , \quad \forall t \in I.$$

The boundedness of sequences expressed in this Corollary implies the convergence of the appropriate subsequences, which by means of a modified version of Ascoli–Arzelà lemma (see [11, 15]) gives

**Lemma 4.8.** *There exists $u \in C(I;H) \cap L^\infty(I;V)$ with $\frac{du}{dt} \in L^\infty(I;V)$ and a subseqence $\{u_{n_k}\}$ of $\{u_n\}$ such that ($\rightharpoonup$ denotes weak convergence)*

$$
\begin{array}{rcll}
u_{n_k} & \to & u \text{ in } C(I;H) & \\
u_{n_k}(t) & \rightharpoonup & u(t) & , \ \forall t \in I, \text{ in } V \\
\bar{u}_{n_k}(t) & \rightharpoonup & u(t) & , \ \forall t \in I, \text{ in } V \\
\frac{du_{n_k}}{dt} & \rightharpoonup & \frac{du}{dt} & , \ \text{ in } L^2(I;H).
\end{array}
$$

Now it remains to prove that the limit $u$ of the sequence of approximations is actually the sought solution of our problem. Note that by its definition $u$ satisfies the equation ( 20 ), i.e., the initial condition. To prove that $u$ satisfies also ( 19 ) we take an arbitrary $v \in V$ and consider convergence of

$$
\int_0^t (\bar{L}_{n_k} \frac{du_{n_k}}{dt}(s), v)ds = \int_0^t (R_n^1(s) + R_n^2(s), v)ds \quad \to \quad \int_0^t ((L_\tau u_t)(s), v)ds (35)
$$

with

$$
R_n^1(s) = (\bar{L}_{n_k} \frac{du_{n_k}}{dt})(s) - (L_\tau \frac{du_{n_k}}{dt})(s)
$$

$$
R_n^2(s) = (L_\tau \frac{du_{n_k}}{dt})(s) - (L_\tau \frac{du}{dt})(s).
$$

We know that $L_\tau$ is continuous over $L^2(I;H)$, moreover, given a sequence of functions $(z_n) \in L^2(I;H)$, such that $z_n(s) \rightharpoonup 0, \forall s \in (0,T)$, we infer that $(L_\tau z_n)(s) \rightharpoonup 0$. Setting

$$
z_n(s) = \frac{du_n}{dt}(s) - \frac{du}{dt}(s)
$$

we get the representation

$$
\begin{aligned}
(R_n^2(s), v) &= ((L_\tau z_{n_k})(s), v) = \int_\Omega \int_0^t \tau(s-r) z_{n_k}(x,r) dr v(x) dx \\
&= \int_0^s \tau(s-r) \int_\Omega z_{n_k}(x,r) v(x) dx dr = \int_0^s \tau(s-r) \left(z_{n_k}(r), v\right) dr
\end{aligned}
$$

and from the last Lemma we get $(R_n^2(s), v) \to 0 \quad , \quad \forall s \in (0,T)$, hence, $\int_0^t (R_2(s), v)ds \to 0$.

It remains to prove $\int_0^t (R_n^1(s), v)ds \to 0$. In turns out that we are able to prove even the strong convergence $R_n^1 \to 0$ in $L^2(I;H)$. Setting $Z(s) = (L_\tau \frac{du_{n_k}}{dt})(s)$, with $Z_i = \frac{1}{h} \int_{I_i} Z(s) ds$ we have $Z \in L^2(I;V)$ and

$$
R_n^1(s) = \frac{1}{h} \int_{I_i} \left[ (L_\tau \frac{du_{n_k}}{dt})(t) - (L_\tau \frac{du_{n_k}}{dt})(s) \right] dt = Z_i - Z(s) \quad , \quad s \in I_i.
$$

By virtue of Lemma ( 3.1 ), there follows $\int_0^T \| \bar{Z}_n(s) - Z(s) \|^2 ds \to 0$ as $n \to \infty$. But $\int_0^T \| R_n^1(s) \|^2 ds = \int_0^T \| \bar{Z}_n(s) - Z(s) \|^2 ds$ and ( 35 ) follows.

Consider now the equation

$$
(\frac{du_n}{dt}(t), v) + \left( (\bar{L}_n \frac{du_n}{dt})(t), v \right) + a(\bar{u}_n(t), v) = (\bar{f}_n(t), v)
$$

which from definition is satisfied by each element of the sequence $(u_n)_{n=1}^{\infty}$. We integrate this equation over $(0,t)$ and get

$$\int_0^t (\frac{du_{n_k}}{dt}(s), v)ds + \int_0^t \left( (\bar{L}_{n_k} \frac{du_{n_k}}{dt})(s), v \right) ds$$
$$+ \int_0^t a(\bar{u}_{n_k}(s), v)ds = \int_0^t (\bar{f}_{n_k}(s), v)ds.$$

Now we pass to the limit with $n_k \to \infty$, using continuity of the form $a(\cdot,\cdot)$, Lebesgue dominated convergence theorem, Lemma ( 3.1 ) and ( 35 ) and obtain

$$\int_0^t (\frac{du}{dt}(s), v)ds + \int_0^t \left( L_\tau \frac{du}{dt})(s), v \right) ds + \int_0^t a(u(s), v)ds$$
$$= \int_0^t (f(s), v)ds \quad , \quad \forall v \in V \quad , \quad \forall t \in I.$$

Differentiate this equation with respect to t and finally obtain

$$(\frac{du}{dt}, v) + (L_\tau \frac{du}{dt}, v) + a(u,v) = (f,v) \quad , \quad \forall v \in V \quad , \quad \text{a.e. } t \in I.$$

This means that the limit function $u$ satisfies equation ( 19 ). Since $u$ satisfies also the initial condition ( 20 ), it is a solution of the problem ( 19 )–( 20 ), which by Lemma ( 4.1 ) is unique.

We note that $V, H$ are reflexive Banach spaces, hence weakly sequentially compact spaces, i.e., (uniform) boundedness of the sequences $\{\bar{u}_n(t)\}$, $\{\frac{du_n}{dt}(t)\}$ in spaces $V$ and $H$ (respectively), together with uniqueness of limits of (convergent) subsequences, imply convergence of the whole sequences.

This gives us finally the main result of this paper.

**Theorem 4.9.** *There exists a unique solution $u$ of the problem ( 19 )-( 20 ), which satisfies $u \in C(I;H) \cap L^\infty(I;V)$ with $\frac{du}{dt} \in L^\infty(I;H)$ and which depends continuously on the data, i.e., the following inequality holds*

$$|u(t)|^2 + \int_0^t \| u(s) \|^2 \leq C \left( |u(0)|^2 + \| f \|^2_{L^2((0,t);L^2(\Omega))} \right). \tag{36}$$

*Proof.* It remains to prove the relation ( 36 ). We consider the equation ( 19 ) integrated with respect to time over $(0,t)$, with $v = u$, which gives

$$\frac{1}{2}|u(t)|^2 - \frac{1}{2}|u(0)|^2 + \int_0^t ((L_\tau u_t)(s), u(s))\, ds + \int_0^t a(u(s), u(s))ds$$
$$= \int_0^t (f(s), u(s))ds.$$

The $\mathcal{LTD}$ property, ellipticity of $a(\cdot,\cdot)$, Cauchy–Schwarz inequality, and Young inequality yield for any $\epsilon > 0$

$$|u(t)|^2 + 2c_a \int_0^t \| u(s) \|^2 \, ds \tag{37}$$
$$\leq (2C_{\mathcal{LTD}} + 1)|u(0)|^2 + \frac{1}{\epsilon}\int_0^t |f(s)|^2 ds + \epsilon \int_0^t |u(s)|^2 ds.$$

Now set $\epsilon = c_a$ and apply Poincaré–Friedrichs ineqiality with the Poincaré constant set to 1 to conclude

$$|u(t)|^2 \leq (2C_{\mathcal{LTD}} + 1)|u(0)|^2 + \frac{1}{c_a} \parallel f \parallel^2_{L^2((0,t),L^2(\Omega))} \; . \tag{38}$$

Combining ( 37 ) and ( 38 ) we get ( 36 ). □

## 5. Acknowledgements

This work is based on research done while the author was on leave at the University of Augsburg and supported by a DAAD fellowship, and it is contained in the author's Ph. D. thesis, written under the gratefully acknowledged scientific care and guidance of Professor Dr. K.-H, Hoffmann.

## References

[1] T. Arbogast, *Analysis of the Simulation of Single Phase Flow Through a Naturally Fractured Reservoir*, SIAM J. Numer. Anal. 26 (1989), pp 12-29

[2] T.Arbogast, J. Douglas, *Dual–Porosity Models for Flow in Naturally Fractured Reservoirs*, Technical Report # 95, Purdue University, April 1989, also : in Dynamics of Fluids in Hierarchical Porous Media, J. H. Cushman, ed., Academic Press, London, 1990, pp. 177–221

[3] T. Arbogast, J. Douglas, and U. Hornung, *Derivation of the Double Porosity Model of Single Phase Flow via Homogenization Theory*, SIAM J. Math. Anal. 21 (1990), pp 823-836

[4] G. I. Barenblatt, Iu. P. Zheltov, and I. N. Kochina, *Basic Concepts in the Theory of Seepage of Homogeneous Liquids in Fissured Rocks (Strata)*, J. Appl. Math. Mech. 24 (1960), pp 1286–1303

[5] A. Bensoussan, J.- L. Lions, G. Papanicolau, *Asymptotic Analysis for Periodic Structures*, North Holland, 1978

[6] H. Gajewski, K. Gröger, K. Zacharias, *Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen*, Akademie–Verlag, Berlin, 1974

[7] G. Gripenberg, S.–O. Londen, O. Staffans, *Volterra Integral and Functional Equations*, Cambridge University Press, 1990

[8] U. Hornung, *Homogenization of Miscible Displacement in Unsaturated Aggregated Soils*, in: Composite Media and Homogenization Theory, G. D. Maso, G. F. Dell'Antonio, eds., Birkhäuser, 1991

[9] U. Hornung, R. E. Showalter, *Diffusion Models for Fractured Media*, J. Math. Anal. Appl., 147 (1990), pp 69-80

[10] J. W. Jerome, *Approximation of Nonlinear Evolution Systems*, Academic Press, 1983

[11] J. Kacur, *Method of Rothe in Evolution Equations*, Teubner, Leipzig, 1985

[12] R.C. MacCamy, J.S. Wong, *Stability Theorems for Some Functional Differential Equations*, Trans. Amer. Math. Soc. 164 (1972), pp 1-37

[13] B. Neta, *Numerical Solution of a Nonlinear Integro–differential Equation*, Jour. of Math. Anal. and Appl. 89 (1982), 598-611

[14] J. W. Nunziato, *On Heat Conduction in Materials with Memory*, Quarterly of Appl. Math., July, 1971

[15] M. Peszyńska, *Fluid Flow Through Fissured Media. Mathematical Analysis and Numerical Approach*, Ph. D. Thesis, University of Augsburg, 1992

[16] M. Peszyńska, *Finite element approximation of diffusion equations with convolution terms*, submitted

[17] E. Sanchez–Palencia, *Non–Homogeneous Media and Vibration Theory*, Lecture Notes in Physics 127, Springer–Verlag, 1980

[18] L.Tartar, *Nonlocal effects induced by homogenization*, in: Essays in Honour of Ennio de Giorgi, F. Colombini et al., eds., Birkhäuser–Verlag, Basel, Switzerland, 1989

Systems Research Institute, Polish Academy od Sciences, ul. Newelska 6 01-447 Warszawa, Poland

*E-mail address*: mpesz@ibspan.waw.pl