ON A MODEL OF NONISOTHERMAL FLOW THROUGH FISSURED MEDIA

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Abstract. A model for nonisothermal flow in double–porosity medium is constructed by the technique of homogenization. The coupling between microscopic and macroscopic equations is discussed, with numerical examples given. The model may be imbedded in a framework common for the models corresponding to different memory effects exhibited by the flow which are related to the hierarchical structure of the medium. For this framework results of numerical experiments are presented.

1. Introduction

In this paper we present a detailed construction of a model describing flow of a fluid through fissured medium in nonisothermal conditions.

In recent years much work has been devoted to modeling, analysis and simulation of single– and multi–component flows in fissured media (see references to this paper). Fissured media are "double–porosity" media composed of a matrix of porous blocks separated by a system of fissures. This hierarchical structure influences the character of the flow which is described by partial differential equations with integral terms. There may occur different geometrical instances of the medium, where the hierarchic structure is developed only in some prescribed direction e.g. in a layered medium (see [10]). Common characteristics for all models is that they take into account several dimension scales: the size of the reservoir, the diameter of porous blocks and the diameter of fissures. The research in microstructure models for aggregated media has been motivated by applications in oil recovery simulation in fractured rock reservoirs ([2, 3, 4, 5, 6, 7]) displacement of contaminated water in aggregated soils ([18, 19, 20]) and additionally by electric charge conduction in multi–layered conductors ([26]).

In this paper we deal with coupled phenomena of mass and energy transfer which occur during flow of a fluid through a fissured medium.

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Such phenomena for the case of an ordinary porous medium have been studied in the context of natural convection (see [13, 14, 16]), multi–component flows where exchange of energy between different components of the mixture occurs (see [12]) or phase changes like evaporation and solidification processes ([17, 23, 25, 33]). Our goal here is to provide an extension of models describing porous media so that the new models would incorporate special characteristics of fissured media but should remain analytically and numerically tractable.

Our model is constructed by the homogenization technique, which enables identification of coefficients and components of the model with the help of the microscopic information about the medium. In section (2) we review general homogenization ideas used in modeling of flows in porous media, which are applied in section (3) to the construction of our model. Technical details of this construction are supplied in the Appendix to this paper. In section (4) we investigate a reduced version of our model which imbeds this model in a framework common for numerous microstructure double–porosity models.

This framework concerns a way of representation of integral delay terms which are related to the double–porosity structure of the medium. When they are represented as convolution integrals, it leads to an explicit decoupling of the macroscopic equations from the microscopic ones. The different kernels of these convolution terms correspond to different types of memory effects exhibited by the flow which can be easily compared in this framework. Another advantage of the convolution representation is that with the help of some special tools developed for the needs of analysis and approximation of integral terms in convolution form (see [28]) we can combine them with what has been done for the models where no hierarchical structure is present. Such an approach was used in the analysis of equations arising from the modeling of diffusion in fissured media (see [28, 30]) as well as in the construction of approximations for the solutions of the models of thermal flows in fissured media in [29]. At the end of section (4) we present results of numerical experiments which show the influence of hierarchical structure of the medium for different delay effects present.

In the following we shall use abbreviations when referring to different models for porous and fissured media, usually denoting the former by PM and the latter by FM, diffusion by D, miscible displacement by MD, nonisothermal flows by NF.

2. Homogenization models of flows in porous and fissured media

In many cases when we are about to model a physical phenomenon, only a partial knowledge of its characteristics is available to us; we are also usually unable to identify the parameters of the underlying mathematical model with physically measurable quantities. In particular we are often given detailed information about a phenomenon available at a certain microscopic level. However a microscopic description though exact is too complex, so we are bound to derive laws governing the phenomenon at a macroscopic level. For the pass from the micro– to the macroscopic level different approaches are possible. For example, for modeling of flows in porous media one may use stochastic and averaging (see [9, 27, 31]) or alternatively homogenization techniques (see [8, 32] for a review) which lead from microscopic description of the fluid motion through channels of the porous body to macroscopic models related to Darcy or Darcy–Forchheimer laws. The advantage of the latter method with respect to the former ones is in a consistent mathematical framework for the analysis of coupling between the micro– and the macrostructure provided by homogenization. In this paper we use homogenization method.

At first we assume that the porous medium has a periodic structure (it is a usual assumption for the homogenization models of porous media) and note that this approach is entirely different from the one used with probabilistic methods (repetitive patterns as opposed to a random structure). In the homogenization setting we start with models P_ϵ describing the phenomenon at a microscopic level, dependent on the period ϵ of the structure. Then we "pass to the limit" with $\epsilon \to 0$ and derive macroscopic models P_* whose solutions u_* correspond to local averages of the microscopic model solutions u_ϵ. The original "exact microscopic" problem which we originally want to study may be regarded as P_ϵ for $\epsilon = 1$ hence the limiting problem solutions u_* provide us with the information about local averages of solutions to the original problem.

The first goal of homogenization is thus the construction of the "limiting problem" P_* whose coefficients are called "effective". This is customarily achieved by means of the multiscale expansion method (see [40]) where the oscillating variables (with frequency related to ϵ) are treated as dependent (spatially) on the two (or more) scales: denoting by x the original space variable we additionally consider the local

variations $y = \frac{x}{\epsilon}$ and obtain

$$u_\epsilon(x) = u_0(x,y) + \epsilon u_1(x,y) + \epsilon^2 u_2(x,y) + \dots \quad , \quad y = \frac{x}{\epsilon} \tag{1}$$

$$\frac{d}{dx_i} = \frac{\partial}{\partial x_i} + \frac{1}{\epsilon}\frac{\partial}{\partial y_i} \tag{2}$$

$$\nabla = \nabla_x + \frac{1}{\epsilon}\nabla_y \tag{3}$$

$$\triangle = \frac{1}{\epsilon^2}\triangle_{yy} + \dots . \tag{4}$$

with u_* denoting the local average (average over the period of the structure) of u_0. The values of macroscopic problem coefficients usually depend explicitly on the microscopic problem coefficients and implicitly on the geometry of the medium via the solution of an auxiliary microscopic problem.

The second goal of homogenization is to prove correctness of the construction, i.e. that as the microscopic structure is refined (i.e. as $\epsilon \to 0$), the sequence of solutions u_ϵ of P_ϵ converges to the solution u_* of P_*.

The "pass" from micro to macro level that we are doing can be seen as an action reverse to "microscope zooming", with ϵ related to the "measure of microscope lenses zooming" adjusted according to the periodic structure of the medium.

The problems P_ϵ and P_* have in general different coefficients and may have a different structure. For example in case when P_ϵ are evolution problems, it is convenient to construct P_* with help of some terms in charge of memory effects related to conservation laws reflected in the microscopic model (see [21, 22, 38, 39]).

Geometry of porous media makes phenomena of flows in porous media an ideal candidate for modeling by homogenization. It is fairly natural to idealize a porous body as a collection of periodically distributed cells, each of them containing a solid and a void cavity part, the first constituting the solid matrix and the second creating the system of channels–pores. Then the steady flow in channels of the porous body is described by Stokes problem defined in each cell's void part and homogenization delivers the macroscopic relation between seepage velocity and pressure gradient which turns out to be equivalent to (experimental) Darcy's law (see [37]) where (static) information about the microstructure is retained in the diffusivity coefficient tensor. This law is a basis on which models describing diffusion in porous media (DPM) are derived. If a more complicated nature of flow is to be described, e.g.

governed at the microscopic level by Navier-Stokes equations, then homogenization leads to macroscopic equations describing the nonisothermal flow in a porous medium (NFPM) (nonisothermal flow in porous media) (see [13, 14]) with some terms reflecting the dynamic influence of the microstructure.

The hierarchical geometry of fissured media requires two "reverse–zooming" steps: the first one in order to pass from the level of an individual pore to the level of a porous body, the second necessary for the pass from the level of a cell of the fissured medium to the level of the whole reservoir. With the double–porosity structure of the medium, viewed as a collection of two porous media of different hydraulic properties (porous matrix and system of fissures), the first step supplies the equations describing the flow inside each of the porous components. In the second step we use individual properties of the porous components (blocks and fissures) calculated as in the first step. The whole two–step procedure is called reiterated homogenization. In a simpler approach we can restrict ourselves to what is supplied by physical measurement at the level of (macro) cells and to phenomenological laws describing both porous components (consistent with what is derived by the first step). In what follows we use the latter approach, i.e. assume that we are given the coefficients of the flow in two porous components of the medium as well as the information about the individual geometry of the cells (in particular proportions between blocks and fissures). With that information we write problems P_f and P_m, with subscripts $_f$ and $_m$ denoting the fissures system and matrix system respectively. The fissured medium is described (as customarily done, e.g. [5]) as a periodic collection of cells, each of which is isometric to a generic cell Ω_0 composed of the block $\Omega_{m,0}$, a part of surrounding fissure $\Omega_{f,0}$ and their interface Γ_0

$$\Omega_0 = \Omega_{f,0} \cup \Omega_{m,0} \cup \Gamma_0$$

Problems P_f and P_m are usually similar in structure but differ in coeffcients. They are complemented by a relation defining behavior of the flow at the interface of the two components, hence, describing the influence of the flow in fissures on that in individual blocks and, reciprocally, contribution of blocks to the total flow in fissures.

For an arbitrary ϵ we (formally) add the $_{,\epsilon}$ subscript to the variables and coefficients and perform some additional scaling necessary to preserve the nature of the flow (the idea of scaling is motivated in the sequel). The problems P_m, P_f with this notation correspond to $P_{f,\epsilon}, P_{m,\epsilon}$ for $\epsilon = 1$. The ϵ–cells may be regarded as images of Ω_0 under translation and shrinking operations $\Omega_0 \Rightarrow \Omega_{0,\epsilon}$ The geometry of

first step

micro		macro
solid and void		blocks and fissures
Stokes	$\Longleftrightarrow$	Darcy's law (DPM)
Navier-Stokes		NFPM

second step

micro		macro
blocks and fissures		double porosity reservoir
DPM	$\Longleftrightarrow$	DFM
NFPM		NFFM

TABLE 1.

ϵ–cells doesn't change under this affine map. The whole domain Ω_ϵ is seen as a collection of ϵ–cells $\Omega_{0,\epsilon}$ spread over the entire reservoir, with $\Omega_{f,\epsilon}$ denoting the totality of fissures system, $\Omega_{m,\epsilon}$ the collection of the blocks (the matrix) and Γ_ϵ the collection of block–fissures interfaces.

Above we have mentioned that the DPM models arise as a result of the first "reverse-zooming" step. In the second step DPM serve as a starting point in construction of models decribing diffusion in fissured media (DFM) for one or more fluid components (miscible and immiscible case) – see [4, 5, 6, 20, 21, 28]. In the case of thermal flows the first step supplies us with NFPM models; as a result of the second step in the sequel NFFM i.e. a model for nonisoithermal flows in fissured media (NFFM)is derived. The construction of the model itself is the first element of the homogenization method. The second one i.e.. the convergence proofs, have been presented for DPM in [37]. The proofs for DFM follow a a promising technique of double–scale convergence as in [1, 7, 24].

The models for porous and fissured media are related to each other as in the table 1.

The remainder of this section is devoted to some homogenization tools used in definition of the coupling between micro and macrostructure. We start with a discussion of "scaling" necessary in definition of ϵ–problems and then we describe the microscopic structure influence on some static and dynamic elements of macroscopic equations: coefficient tensors and kernels of integral memory terms, respectively.

Scaling in homogenization

Here we motivate the "scaling"in ϵ–problems. Let us consider diffusion in a medium composed of n regularly distributed square cells built of circular porous blocks isometric to a generic block Ω_0 of radius R_1 separated by a system of channels. Let us assume that the porous blocks are initially filled with some fluid and the channels are initially void. For simplicity we consider two–dimensional domains of flow. If we consider the ϵ–geometry of the domain, we'll be given $\epsilon^{-2}n$ cells $\Omega_{i,\epsilon}$, with each of circular blocks of radius $R_\epsilon = \epsilon^2 R_1$. The diffusion in each block is described by the following system of equations

$$\begin{aligned} \frac{\partial u_\epsilon}{\partial t} &= c_\epsilon \triangle u_\epsilon \text{ in } \Omega_{i,\epsilon} \\ u_\epsilon &= 0 \text{ on } \partial\Omega_{i,\epsilon}, t > 0 \\ u_\epsilon(\cdot, 0) &\equiv 1 \text{ in } \Omega_{i,\epsilon} \end{aligned}$$

satisfied in each block $\Omega_{i,\epsilon}$ with $i = 1...n\epsilon^{-2}$ isometric to $\Omega_{0,\epsilon}$. The outflowing flux from a block for $t > 0$ (into the system of channels) is given by the divergence theorem as

$$Q_{i,\epsilon}(t) = \int_{\Omega_{i_\epsilon}} \frac{\partial u_\epsilon}{\partial t}(y,t)dy$$

while its average is $q_{i,\epsilon} = \frac{1}{|\Omega_{i,\epsilon}|} Q_{i,\epsilon}(t)$ and the total flux by isometry is equal to

$$T_\epsilon(t) = n\epsilon^{-2} \int_{\Omega_{0_\epsilon}} \frac{\partial u_\epsilon}{\partial t}(y,t)dy \quad , \quad t > 0$$

It is now our goal to demonstrate that in order to prevent the total flow T_ϵ from attaining an unjustified growth (it is because we want to preserve the character of the flow), one has to scale the diffusion coefficient c_ϵ as the blocks shrink (i.e. as $R_\epsilon \to 0$). This is clear from the results of numerical experiments presented in the table (2) (see also heuristic reasoning in [5]). The values in the table have been calculated by means of the Galerkin approximation method combined with implicit Euler scheme for discretization in time. As ϵ decreases, the value of T_ϵ grows, unless the diffusion coeffcient c_ϵ is appropriately scaled by a factor ϵ^2.

On the other hand, scaling may be justified by purely technical arguments. If we use the multiple scales expansion technique for asymptotic expansion of ϵ–variables and do not scale the diffusion coefficients, some terms of the asymptotic expansion will be dropped during calculation of coeffcients. For example, let us consider the Stokes problem defined

#	ϵ	c_ϵ	$\epsilon^{-2}c_\epsilon$	$Q_{0,\epsilon}$	T_ϵ	$q_{0,\epsilon}$
1	1	1	1	28.9564	28.9564	9.23195
2	0.5	1	4	18.3304	73.3216	23.3765
3	0.5	0.25	1	7.23911	28.95644	9.23195
4	0.1	1	100	2.31633	231.633	73.8497
5	0.1	0.01	1	0.289564	28.9564	9.23193

TABLE 2. Flow quantities with respect to parameters ϵ, c_ϵ

for void ϵ–blocks $\Omega_{i,\epsilon}$ separated from the solid by the interface $\partial\Omega_{i,\epsilon}$

$$\begin{aligned} \nabla p_\epsilon - k_\epsilon \triangle \bar{v}_\epsilon &= 0 \ , \quad y \in \Omega_{i,\epsilon} \\ \nabla \cdot \bar{v}_\epsilon &= 0 \ , \quad y \in \Omega_{i,\epsilon} \\ \bar{v}_\epsilon &= 0 \ , \quad y \in \partial\Omega_{i,\epsilon} \end{aligned}$$

Using the expansion $\bar{v}_\epsilon(x) = \bar{v}_0(x,y)+\epsilon\bar{v}_1(x,y)+\epsilon^2\bar{v}_2(x,y)+\ldots$ and the formula for derivatives (see above e.g (3)) we arrive at a conclusion (comparing terms at like powers of ϵ) that if k_ϵ does not depend on ϵ, then the terms satisfy $\bar{v}_0 \equiv \bar{v}_1 \equiv 0$. It means that the asymptotic expansion might as well start with the second term $\bar{v}_2$. But, if the scaling is $k_\epsilon = \epsilon^2 k_1$ is present, the first terms of the expansion are retained.

Effects of microstructure on macroscopic quantities.

Let us be given a generic cell $C \subset R^d, d = 2, 3$ with a connected subset C_{out} adjacent to the cell's boundary ∂C, with interface Γ separating C_{out} from its complement C_{in} in C . Then we define $\omega_j^C, j = 1, ...d$ as the C–periodic functions (determined up to an additive constant) which are harmonic on C_{out} and satisfy flux boundary conditions prescribed by unit vectors $\bar{e}_j \in R^d, j = 1, ...d$ on the interface Γ, precisely,

$$\begin{aligned} \nabla_y^2 \omega_j &= 0 \ , \quad y \in C_{out} \\ \nabla_y \omega_j \times \eta &= -\bar{e}_j \times \eta \ , \quad s \in \Gamma. \end{aligned}$$

We define also tensor $\sigma_{i,j}^C, i, j = 1..d$ of elements

$$\sigma_{ij}^C = \frac{1}{|C_{out}|} \int_{C_{out}} \frac{\partial \omega_j}{\partial y_i} dy.$$

This tensor is used for calculation of the effective diffusivity (in general, coefficient of the laplacian) tensor. In the first reverse–zooming step (Darcy's law derived from microscopic formulation of Stokes law) values of diffusivity are equal to values of σ^C scaled by the original diffusivity coefficient. In case of the second reverse–zooming step (diffusion in

#	R_{in}	$diam(C)$	$\sigma^C_{1,1}$	$\sigma^C_{1,2}$	$\sigma^C_{2,1}$	$\sigma^C_{2,2}$
1	0.1	1	0.0217545	-3.08218e-10	-3.29225e-10	0.0217545
2	0.3	1	0.144773	4.79794e-08	4.39782e-08	0.144773
3	0.45	1	0.116305	5.47182e-09	8.99997e-09	0.116305
4	0.2	2	0.0870183	9.14788e-09	9.23737e-09	0.0870183
5	0.6	2	0.579091	1.83346e-07	1.97297e-07	0.579091
6	0.9	2	0.465218	-3.46946e-07	1.72928e-07	0.465219

TABLE 3. Numerically calculated values of σ^C

fissured media) the diffusivity coefficient arises as the sum of $\sigma^C_{ij} + \delta^K_{ij}$ (where δ^K is Kronecker's symbol) scaled as above and additionally multiplied by a scalar coefficient reflecting proportions between blocks and fissures (see [5, 6] or derivation of NFFM in the sequel).

The values of ω^C hence also of σ^C depend on the geometry of the original generic cell. In order to calculate their numerical approximation one needs to impose supplementary conditions to ensure uniqueness of the solution (hence solvability of the approximation procedure). Sample values obtained for a square cell C of diameter $diam(C)$ with circle shaped C_{in} (of radius R_{in}) are presented in the table (3).

These results in particular show how the proportions between the size of C_{out} and of its complement C_{in} (rows 1, 2, 3, and 4, 5,6) influence calculated values of the tensor σ^C. On the other hand, we observe that as the cell's size increases with ϵ, the corresponding values of σ^C change with ϵ^2. For example, compare values in rows 1 and 4: the size of the cell in 4'th row is doubled with respect to the 1'st row, and the values of tensors are exactly scaled by factor 4 which is consistent with the "scaling" issue discussed above. Hence, the actual size of the generic cell (for $\epsilon = 1$) is essential. The smaller the size of the generic cell, the less significant is the contribution of σ^C to the macroscopic diffusivity coefficient (recall that σ^C_{ij} are added to Kronecker's symbol (identity matrix), hence, the microstructure's influence is less significant.

The influence of the geometry and coefficients given on the microscale is also evident when considering the contruction of the evolution problems. We note that the tensor σ^C providing a contribution to diffusion coefficients depends on the "static microstructure" of the problem.

Now we shall consider an evolution problem defined in C_{in} with prescribed values on the interface Γ.

$$\begin{aligned} \frac{\partial v}{\partial t} - k \triangle v &= 0 \ , \quad y \in C_{in} \\ v|_\Gamma &\equiv b(t) \ , \quad t > 0 \\ v(y,0) &= a(y) \equiv const \ , \quad y \in C_{in} \end{aligned}$$

We define $r, \tilde{r}$ to be the solutions of the above problem for $b \equiv 0, a = 1$ and $b \equiv 1, a = 0$, respectively. Then the function v (for general boundary and initial conditions) can by means of Green function representation be written as

$$v(y,t) = ar(y,t) + \int_0^t b(s) \frac{\partial \tilde{r}}{\partial t}(y, t-s) ds = ar(y,t) + (b \star \frac{\partial \tilde{r}}{\partial t})(y,t)$$

Define a function $\tau^{C,k}(t) = -\frac{d}{dt}\frac{1}{|C_{in}|}\int_{C_{in}} r(y,t)dy$, note that since $r + \tilde{r} \equiv 1$ we have $\tau^{C,k} = \frac{d}{dt}\frac{1}{|C_{in}|}\int_{C_{in}} \tilde{r}(y,t)dy$ and calculate the flux $Q^v(t)$ flowing out of C_{in}

$$\begin{aligned} Q^v(t) &= \int_\Gamma k(\nabla v \cdot \eta)(s,t) ds = \int_{C_{in}} \frac{\partial v}{\partial t}(y,t) dy \\ &= \int_{C_{in}} \left\{ a \frac{\partial r}{\partial t} + \frac{\partial}{\partial t}\left[b \star \frac{\partial \tilde{r}}{\partial t} \right] \right\} dy = \\ &= -\left(-a|C_{in}| \frac{d}{dt}\left[\frac{1}{|C_{in}|} \int_{c_{in}} r(y,t) dy \right] \right) + \int_{C_{in}} \left[b(0) \frac{\partial \tilde{r}}{\partial t} + \left(\frac{\partial b}{\partial t} \star \frac{\partial \tilde{r}}{\partial t} \right) \right] \\ &= -|C_{in}| \left[\tau^{C,k} \star \left(\frac{\partial b}{\partial t} + \delta(b(0) - a) \right) \right] \end{aligned}$$

where we have used derivation rules for the convolution product and the fact that δ (Dirac symbol) is the unity of this product.

We note that the expression $b(0) - a$ is a measure of "incompatibility" between initial and boundary condition which is instantaneously propagated (in most models initial "equilibrium" conditions are assumed, hence, $b(0) = a$), in contrast to the diffusion influenced by boundary flux $\frac{\partial b}{\partial t}$ which is delayed with "speed" controlled by the function $\tau^{C,k}(\cdot)$. Function $\tau^{C,k}$ is in general singular at zero (see its definition), positive, and monotone decreasing, sice it describes dynamics of a diffusion problem with no sources. Its values strongly depend on the geometry of C and on the coefficient k and can be calculated only in an approximate way. In some simple cases ($k \equiv 1$, trivial geometry of the cell C e.g. circular block C_{in} embedded in a rectangular C) one can calculate $\tau^{C,k}(\cdot)$ analytically as well as numerically and then a relatively good agreement of respective values is obtained (see [28]).

In summary, the values of the convolution kernel $\tau^{C,k}$ and auxiliary diffusivity tensor σ^C are dependent on the microstructure. In the sequel we shall show how they are used in construction of the model NFFM.

3. Construction of NFFM

In this section we present homogenization construction of the model NFFM describing flow of a thermally dilatable fluid (liquid) in a fissured medium according to the second "reverse-zooming" step explained above. We start from a model describing the nonisothermal flow in an ordinary porous medium (NFPM), then we shall write down equations constituting P_m, P_f, i.e., the versions of NFPM for both components of the hierarchic structure. Next we define their ϵ–versions (with appropriate scaling) and pass to the limit in order to get P_*. We follow the construction principles developed in [5, 6, 37] and refer to these papers for the details.

The model NFPM we use is based upon the models presented in [12, 13, 14, 16] and is composed of three equations describing conservation of momentum, mass and energy. The flow is defined in $\mathcal{Q} = \Omega \times I$ with $I = (0, T), T > 0$, where $\Omega \subset R^d, d = 2, 3$ is an open bounded domain with a smooth boundary $\partial\Omega$. We'll be seeking values of temperature θ, velocity $\bar{v}$, and pressure p of a dilatable Newtonian fluid (liquid) whose density ρ and viscosity μ are governed by the constitutive equations

$$\rho(\theta) = \rho_{ref}(1 - \beta(\theta - \theta_{ref})) \tag{5}$$

$$\mu(\theta) = \mu_{ref} e^{-\gamma(\theta - \theta_{ref})} \tag{6}$$

in which $\beta, \gamma, \rho_{ref}, \mu_{ref} > 0$. The (porous) medium in which the flow occurs is characterized by coefficients of permeability K and porosity α. The specific heat capacity of the fluid is denoted by c_{fl} and that of the solid's by c_{sol}. The fluid is of thermal conductivity λ (for simplicity we consider λ constant, but the model might be easily modified in order to account for the temperature dependence) and heat capacity $c(\theta) = c_{fl}\rho(\theta)$. The total heat capacity of the ensemble of fluid contained in the medium is equal

$$b(\theta) = \rho_s c_{sol}(1 - \alpha) + c(\theta)\alpha \tag{7}$$

We are also given the gravity acceleration vector $\bar{g}$ which enters into the conservation of momentum equation assumed in Darcy's form as

$$\mu(\theta)\bar{v} + K\left(\nabla p + \rho\bar{g}\right) = 0 \ , \quad \text{in } \mathcal{Q}. \tag{8}$$

Alternatively one might consider the momentum conservation laws in Forchheimer or Navier-Stokes form. However in what follows we deal

with fissured media where it is customary to assume laminarity of the filtration occuring in both components of the medium (see [5]).

The continuity equation is

$$\alpha \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho(\theta)\bar{v}) \;=\; 0 \;, \quad \text{in } \mathcal{Q}. \tag{9}$$

In some cases it is convenient to rewrite this equation using (5) and $A = -\alpha\beta$ as

$$A \frac{\partial \theta}{\partial t} + \nabla \cdot (\rho(\theta)\bar{v}) \;=\; 0 \;, \quad \text{in } \mathcal{Q}. \tag{10}$$

the energy equation is of the form

$$b(\theta) \frac{\partial \theta}{\partial t} - \nabla \cdot (\lambda \nabla \theta) + c(\theta)\bar{v} \cdot \nabla \theta \;=\; 0 \;, \quad \text{in } \mathcal{Q}. \tag{11}$$

The above conservation (8), (10), (11) and constitutive laws (5), (6) when complemented by suitable boundary (for pressure, temperature and velocity) and initial conditions (for temperature) constitute a well–posed model (see [16]). This model will now be rewritten for the description of P_ϵ composed of $P_{f,\epsilon}, P_{m,\epsilon}$, I_ϵ i.e. of the ϵ–problems defined in fissures and in the matrix with a suitable interface problem, respectively. The interface conditions in I_ϵ are of the "constant approximation" type (see [11] for general framework of interface conditions) i.e. they correspond to interaction between blocks and fissures only through diffusion. We note that appropriate scaling is used, with factor ϵ^2 for diffusion coefficients in matrix problems and ϵ^{-1} in order to preserve gravity terms in the matrix equations. See [2] for a discussion of scaling of gravity terms and [5] for general scaling heuristics..

We have the following system:

fissure problem $P_{f,\epsilon}$ (in $\Omega_{f,\epsilon} \times I$)

$$A_f \frac{\partial \rho_{f,\epsilon}}{\partial t} + \nabla \cdot (\rho(\theta_{f,\epsilon})\bar{v}_{f,\epsilon}) = 0 \tag{12}$$

$$\mu(\theta_{f,\epsilon})\bar{v}_{f,\epsilon} + K_f \left(\nabla p_{f,\epsilon} + \rho_{f,\epsilon}\bar{g}\right) = 0 \tag{13}$$

$$b_f(\theta_{f,\epsilon}) \frac{\partial \theta_{f,\epsilon}}{\partial t} - \nabla \cdot (\lambda_{f,\epsilon} \nabla \theta_{f,\epsilon}) + c_f(\theta_{f,\epsilon})\bar{v}_{f,\epsilon} \cdot \nabla \theta_{f,\epsilon} = 0, \tag{14}$$

matrix problem $P_{m,\epsilon}$ (in $\Omega_{m,\epsilon} \times I$)

$$\alpha_m \frac{\partial \rho_{m,\epsilon}}{\partial t} + \epsilon \nabla \cdot (\rho(\theta_{m,\epsilon}) \bar{v}_{m,\epsilon}) = 0 \quad (15)$$

$$\mu(\theta_{m,\epsilon}) \bar{v}_{m,\epsilon} + \epsilon K_m \left(\nabla p_{m,\epsilon} + \frac{1}{\epsilon} \rho_{m,\epsilon} \bar{g} \right) = 0 \quad (16)$$

$$b_m(\theta_{m,\epsilon}) \frac{\partial \theta_{m,\epsilon}}{\partial t} - \epsilon^2 \nabla \cdot (\lambda_{m,\epsilon} \nabla \theta_{m,\epsilon}) + \epsilon c_m(\theta_{m,\epsilon}) \bar{v}_{m,\epsilon} \cdot \nabla \theta_{m,\epsilon} = 0 \quad (17)$$

interface problem I_ϵ (on $\Gamma_\epsilon \times I$)

$$p_{m,\epsilon} = p_{f,\epsilon} \quad (18)$$

$$\theta_{m,\epsilon} = \theta_{f,\epsilon} \quad (19)$$

$$\epsilon \bar{v}_{m,\epsilon} = \bar{v}_{f,\epsilon} \quad (20)$$

$$\epsilon^2 \lambda_{m,\epsilon} \nabla \theta_{m,\epsilon} \cdot \eta = \lambda_{f,\epsilon} \nabla \theta_{f,\epsilon} \cdot \eta \quad (21)$$

Our goal now is to determine the equations governing the flow at the macroscopic level. We shall use the two–scale asymptotic expansion method (see section (2)) and represent the unknowns describing the flow as dependent (spacially) on the "slow scale" corresponding to the location of the cell in the entire reservoir, and on the "fast scale" which is related to the location of the point within the particular cell (see also the approach in distributed models for fissured media which is (unformally) consistent with this method e.g. [34, 35, 36]). For $\bullet$ denoting $_f$ or $_m$ we pass from $x \in \Omega_{\bullet,\epsilon}$ to $(x, y) \in \Omega \times \Omega_0$.

$$\theta_{\bullet,\epsilon}(x) = \theta_{\bullet,0}(x, y) + \epsilon \theta_{\bullet,1}(x, y) + \epsilon^2 \theta_{\bullet,2}(x, y) + \cdots \quad (22)$$

$$p_{\bullet,\epsilon}(x) = p_{\bullet,0}(x, y) + \epsilon p_{\bullet,1}(x, y) + \epsilon^2 p_{\bullet,2}(x, y) + \cdots \quad (23)$$

$$\bar{v}_{\bullet,\epsilon}(x) = \bar{v}_{\bullet,0}(x, y) + \epsilon \bar{v}_{\bullet,1}(x, y) + \epsilon^2 \bar{v}_{\bullet,2}(x, y) + \cdots \quad (24)$$

We also use Taylor expansion for solution dependent coefficients

$$f(\theta_{\bullet,\epsilon}) = f(\theta_{\bullet,0}) + \epsilon f_{\bullet,1} + \epsilon^2 f_{\bullet,2} + O(\epsilon^3)$$

To obtain the limiting problem P_* we insert the asymptotic relations together with consistent rules for derivation (like (3)) into the equations constituting P_ϵ. We then use averaging over Ω_0 and finally get P_* whose solutions are the first terms of asymptotic expansions. Problem P_* is composed of the macroscopic problem $P_{f,*}$ defined for $x \in \Omega$ and of $P_{m,*}$ defined for $y \in \Omega_0$. These two problems are coupled by interface conditions I_* defined for $s \in \Gamma_0$ and by the flux memory terms L_1, L_2

defined as follows:

$$L_1 = \frac{1}{\Omega_0}\int_{\Omega_{m,0}} \alpha_m \frac{\partial \theta_{m,0}}{\partial t} dy \tag{25}$$

$$L_2 = \frac{1}{\Omega_0}\int_{\Omega_{m,0}} \left(b_m(\theta_{m,0})\frac{\partial \theta_{m,0}}{\partial t} + c(\theta_{m,0})\bar{v}_{m,0}\cdot\nabla_y\theta_{m,0}\right) dy \tag{26}$$

Problem $P_{f,*}$ solved for $(\theta_{f,0}, p_{f,0}, \bar{V}_0)$ in $\mathcal{Q}$. with $\bar{V}_0 = \frac{1}{\Omega_0}\int_{\Omega_{f,0}} \bar{v}_{f,0}(y)dy$

$$\alpha_* \frac{\partial \rho_{f,0}}{\partial t} + \nabla\cdot(\rho(\theta_{f,0})\bar{V}_0) = -L_1 \tag{27}$$

$$\mu(\theta_{f,0})\bar{V}_0 + K_*\left(\nabla p_{f,0} + \rho_{f,0}\bar{g}\right) = 0 \tag{28}$$

$$b_*(\theta_{f,0})\frac{\partial \theta_{f,0}}{\partial t} - \nabla\cdot(\lambda_*\nabla\theta_{f,0}) + c_*(\theta_{f,0})\bar{V}_0\cdot\nabla\theta_{f,0} = -L_2 \tag{29}$$

Problem $P_{m,*}$ solved for $(\theta_{m,0}, p_{m,0}, \bar{v}_{m,0})$ in $\Omega_{m,0}\times I$

$$\alpha_m \frac{\partial \rho_{m.0}}{\partial t} + \nabla\cdot(\rho(\theta_{m,0})\bar{v}_{m,0}) = 0 \tag{30}$$

$$\mu(\theta_{m,0})\bar{v}_{m,0} + K_m\left(\nabla p_{m,0} + \rho_{m,0}\bar{g}\right) = 0 \tag{31}$$

$$b_m(\theta_{m,0})\frac{\partial \theta_{m,0}}{\partial t} - \nabla\cdot(\lambda_m\nabla\theta_{m,0}) + c_m(\theta_{m,0})\bar{v}_{m,0}\cdot\nabla\theta_{m,0} = 0. \tag{32}$$

Problem I_* on $\Gamma_0\times I$

$$\theta_{m,0} = \theta_{f,0} \tag{33}$$

$$p_{m,0} = p_{f,0}. \tag{34}$$

Boundary and initial conditions

$$\lambda\nabla\theta\cdot\bar{\eta} = 0\ , \quad \text{on } \partial\Omega\times I \tag{35}$$

$$\bar{v}\cdot\bar{\eta} = 0\ , \quad \text{on } \partial\Omega\times I \tag{36}$$

$$\theta(\cdot,0) = \theta_0\ , \quad \text{in } \Omega \tag{37}$$

This is the full (exact) version of the limiting problem P_* describing the nonisothermal flow in a fissured medium (NFFM). We note that it is structurally similar to NFPM. The basic difference, except for the new (effective) coefficients is in the distributed source (density and energy)terms on the right hand side of (29) and (27). These terms give rise to memory effects, because they describe mutual interaction of diffusion in blocks and in fissures: in particular the response of the blocks to the flow in surrounding fissures affected by values of quantities describing the flow local to blocks. These response effects are distributed over the whole reservoir.

Now we discuss the structure of NFFS with respect to the structure of models describing miscible flows in porous and fissured media (miscible displacement: MDPM, MDFM, see [3, 5, 15, 4]). There is a significant

similarity between NFPM and MDPM present in the microscopic models (i.e. the models for porous media) which is preserved when passing to modeling of flows in hierarchical porous media (MDFM and NFFM). In MDPM we deal essentially with three component equations of the model which describe conservation of mass and of momentum as well as the balance of mixed concentrations. This structure is similar to the present case, with the concentration equation MD replaced in our case by the energy conservation equation. Additionally, the memory terms in MDFM and NFFM are of a similar form.

However, in DFFM we deal with coupled phenomena of energy and mass transport, while miscible displacement models describe only the latter phenomenon. Another essential difference is that of the character of the flow being described. In MD one has to deal with sharp concentration fronts and viscous fingering effects that must be captured by an appropriate numerical procedure. In case of thermal flows in a broad range of temperature variations the dynamics of changes is less vivid; however the nonhomogenities of the medium may concern thermal properties as well as hydraulical ones and consequently NFFM require some care in analysis and approximation.

In the end let us discuss how the tools developed for the models of porous media carry over to the models of fissured media. Let us assume that we are provided with a method suitable for analytic and numerical treatment of a model describing certain phenomenon arising in porous media. In general we are not guaranteed that the same method applies to an analogous model describing the same phenomenon in fissured media. In case of diffusion i.e. DPM and DFM models, the appplication of an appropriate semi–discrete approximation method allows for the consistent treatment of both models (see [28]). In the present case, i.e., of models NFPM and NFFM, it turns out that if we reduce the generality of the model NFFM, then the tools developed for the needs of DFM and NFPM can be combined in order to formulate a convergent approximation scheme appropriate for NFFM. In the next section we propose a suitable reduction in generality of the model NFFM.

4. Reduced model

In the previous section we showed the construction of a model NFFM which described the flow of a fluid through a fissured medium in nonisothermal conditions. The purpose of the present section is to show how one can restrict the generality of the model NFFM in order to get a model which is still adequate for description of a broad subclass of

problems and which is advantageous from the point of view of analysis and approximation.

The reduction of the model concerns the microscopic problem (30)–(32), which in our opinion doesn't need to be as general as originally stated. The microscopic problem is defined in a representative block i.e. in the part where most of the storage of fluid takes place. On the other hand the block is the "slow flow" part of the cell: diffusion, convection and other transport phenomena are very slow with respect to what is happening in fissures. We propose to ignore the convection phenomena (the convection term in the energy equation), because the velocity values (driven by pressure changes) and the temperature variations OK -0 I'm baxck - window problems - are very small with respect to the other components of this equation. On the other hand, if one wants to account for any first-order terms in the block equations, then the interface conditions should be necessarily modified. Since in this paper we assumed the approximation–by–constant form of interface coupling, it is consistent to drop the first-order terms in the blocks equation.

Another simplification of the model arises from a practical observation. In case of blocks, the value of coeffcient $b(\cdot)$ in the microscopic energy equation is parametrized by porosity of blocks (see (7)): this coefficient is the sum of 2 terms, the first one corresponding to the fluid part and the other to the solid part of the medium. Explicit calculation shows that for blocks, the proportion between the two components of this sum doesn't change much with the fluid's temperature variations because the solid fraction dominates the ensemble and the solid's density is independent of the temperature. This feature is relative to what is happening in fissures : the variations of $b(\cdot)$ in fissures with respect to blocks are proprotional to the ratio between porosities of the porous medium corresponding to fissures and blocks. This ratio may be typically of an order of magnitude. This reasoning shows that with not much error one can approximate $b(\cdot)$ by a constant $\tilde{b}$, which leads to some considerable simplifications in the model.

Taking into account the proposed simplifications (convection terms dropped, constant approximation $\tilde{b}$ of b), we rewrite the microscopic energy equation (which is now decoupled from the other equations) in the form

$$\tilde{b}_m \frac{\partial \theta_{m,0}}{\partial t} \quad - \quad \nabla \cdot (\lambda_m \nabla \theta_{m,0}) = 0. \tag{38}$$

The microscopic variable $\theta_{m,0}$ after this modification is a solution of a linear equation. Hence, the flux terms L_1, L_2 in the macroscopic energy equation can be rewritten according to convolution representation introduced in the section (2). Precisely, we define two convolution kernels τ_1, τ_2 with τ_2 related to solution of (38) with appropriate unit initial conditions and homogeneous Dirichlet condition imposed on the interface Γ_0. This construction follows the one of $\tau^{C,k}$ in section (2) with $C = \Omega_0, C_{in} = \Omega_{m,0}, k = \frac{\lambda_m}{b_m}$. The kernel τ_1 is defined analogously, with $k = \frac{1}{A_m}$. The flux terms L_1, L_2 can now be represented as

$$Ł_i(t, v) = (\tau_i \star v)(t) = \int_0^t \tau_i(t-s)v(s)ds \ , \quad t \in I, i = 1, 2, v \in L^2(I) \tag{39}$$

i.e. the macroscopic equations are decoupled from explicit dependence on the microstructure. The values of coefficients as well as of convolution kernels can be calculated once and be reused in all time steps, when necessary. Memory effects exhibited by the flow are directly dependent on the flow dynamics and indirectly on the microstructure dependent coefficients and function kernels.

We note that the convolution representation provides a common framework for different types of dynamics of flow in blocks which is equivalent to different delays in response of the blocks to local variations in surrounding fissures (see also [18]). The steady flow in blocks correponds to a nonsingular convolution kernel, and at the two extremes we have convolution kernels corresponding to "no–flow" and "dynamic flow" in blocks. The latter is as introduced above, the former one corresponds to the case when blocks are not porous but impermeable hence the whole fissured medium can be treated as an ordinary porous medium (it is the case of an ordinary diffusion equation with no delay terms i.e. "instantaneous reaction"). The figures 1,2 contain results of a numerical experiment showing the kernels and their influence on the macroscopic solution. In the figure 1 there are the three curves drawn which correspond to three different convolution kernels discussed above: the $\tau = 0$ (instantaneous response: "no–flow" in blocks case), nonsingular ("slow–flow" case), and a singular function $\tau(\cdot)$ (its values are calculated for the case of a circular block imbedded in a rectangular cell and approximated numerically, see section (2)). In the figure 2 the three curves (corresponding to the respective kernels) represent evolution in time of solution of the problem (temperature) corresponding to different kernels, with an external source present. One notes an essential difference between graphs corresponding to the singular and nonsingular cases, and a similarity of the "slow–flow" to the "no–flow" case.

5. Appendix

This section contains the technical part of the construction of the model NFFM by homogenization method, i.e., derivation of P_* from P_ϵ. It is a supplement to the content of section (3), explaining how one derives $P_{f,*}, P_{m,*}, I_*$ from $P_{f,\epsilon}, P_{m,\epsilon}, I_\epsilon$. Below we use notation and relations introduced in section (3).

First we insert (22), (24) and derivation rules into (10) to get

$$\begin{aligned} A_f\frac{\partial\theta_{f,0}}{\partial t} &+ \epsilon A_f\frac{\partial\theta_{f,1}}{\partial t} + \epsilon^2 A_f\frac{\partial\theta_{f,0}}{\partial t} \\ +\nabla_x\cdot[\rho_f(\theta_{f,0})\bar v_{f,0} &+ \epsilon\left(\rho_f(\theta_{f,0})\bar v_{f,1}+\rho_{f,1}\bar v_{f,0}\right)] + \\ +\nabla_y\cdot\Big[\epsilon^{-1}\rho_f(\theta_{f,0})\ \bar v_{f,0} &+ (\rho_f(\theta_{f,0})\bar v_{f,1}+\rho_{f,1}\bar v_{f,0}) + \epsilon(\rho_f(\theta_{f,0})\bar v_{f,2} \\ &+ \rho_{f,1}\bar v_{f,1}+\rho_{f,2}\bar v_{f,0})] + \mathcal{O}(\epsilon^2) = 0 \end{aligned}$$

Since this equation must be satisfied for all ϵ, the sum of terms of equal powers of ϵ should vanish. Hence for powers of ϵ of order $-1, 0, 1$ we obtain respectively

$$\nabla_y\cdot[\rho_f(\theta_{f,0})\bar v_{f,0}] = 0 \tag{40}$$

$$\begin{aligned} A_f\frac{\partial\theta_{f,0}}{\partial t}+\nabla_x\cdot[\rho(\theta_{f,0})\bar v_{f,0}] &+ \nabla_y\cdot[\rho_f(\theta_{f,0})\bar v_{f,0}] \\ +\nabla_y\cdot\rho_f(\theta_{f,0})\bar v_{f,1} &+ \rho_{f,1}\bar v_{f,0}] = 0 \end{aligned} \tag{41}$$

$$\begin{aligned} A_f\frac{\partial\theta_{f,1}}{\partial t} &+ \nabla_x\left[(\rho_f(\theta_{f,0})\bar v_{f,1}\rho_{f,1}\bar v_{f,0})\right] \\ &+ \nabla_y\cdot[\rho(\theta_{f,0})\bar v_{f,2}\rho_{f,1}\bar v_{f,1} - \rho_{f,2}\bar v_{f,0}] = 0 \end{aligned} \tag{42}$$

With this technique we rewrite all model component equations and compare equal powers of ϵ to get for the different parts of the model the following relations

Problem I_ϵ: 0,1 powers

$$p_{m,0} = p_{f,0} \tag{43}$$

$$p_{m,1} = p_{f,1} \tag{44}$$

$$\theta_{m,0} = \theta_{f,0} \tag{45}$$

$$\bar v_{f,0}\cdot\eta = 0 \tag{46}$$

$$\bar v_{f,1}\cdot\eta = \bar v_{m,0}\cdot\eta \tag{47}$$

$$\bar v_{f,2}\cdot\eta = \bar v_{m,2}\cdot\eta \tag{48}$$

$$(\lambda_f\nabla_x\theta_{f,0}+\lambda_f\nabla_y\theta_{f,1})\cdot\eta = 0 \tag{49}$$

$$\lambda_f\nabla_y\theta_{f,0}\cdot\eta = 0 \tag{50}$$

$$\lambda_f\nabla_x\theta_{f,1}\cdot\eta+\lambda_f\nabla_y\theta_{f,2}\cdot\eta = \lambda_m\nabla_y\theta_{m,0}\cdot\eta \tag{51}$$

Momentum equation for fissures: -1,0 powers

$$K_f \nabla_y p_{f,0} = 0 \quad (52)$$

$$\mu(\theta_{f,0})\bar{v}_{f,0} + K_f\left[\nabla_x p_{f,0} + \rho(\theta_{f,0})\bar{g}\right] + K_f \nabla_y p_{f,1} = 0 \quad (53)$$

Energy equation for fissures: -2,-1,0 powers

$$\nabla_y \cdot (\lambda_f \nabla_y \theta_{f,0}) = 0 \quad (54)$$

$$\begin{aligned} \nabla_x \cdot (\lambda_f \nabla_y \theta_{f,0}) &+ \nabla_y \cdot (\lambda_f \nabla_x \theta_{f,0}) \\ +\nabla_y \cdot (\lambda_f \nabla_y \theta_{f,1}) &+ +c(\theta_{f,0})\bar{v}_{f,0} \cdot \nabla_y \theta_{f,0} = 0 \end{aligned} \quad (55)$$

$$\begin{aligned} b_f(\theta_{f,0})\frac{\partial \theta_{f,0}}{\partial t} &- \nabla_y(\lambda_f \nabla_y \theta_{f,2}) \\ -\nabla_x(\lambda_f \nabla_x \theta_{f,0}) &- \nabla_x(\lambda_f \nabla_y \theta_{f,1}) - \nabla_y(\lambda_f \nabla_x \theta_{f,1}) \\ +c(\theta_{f,0})\bar{v}_{f,0} \cdot \nabla_x \theta_{f,0} &+ c(\theta_{f,0})\bar{v}_{f,0} \cdot \nabla_y \theta_{f,1} + c(\theta_{f,0})\bar{v}_{f,1} \cdot \nabla_y \theta_{f,0} \\ &+ c_1 \bar{v}_{f,0} \cdot \nabla_y \theta_{f,0} = 0 \end{aligned} \quad (56)$$

Now we remark that from (52) it follows that $p_{f,0}$ is a function of x only. Similar corollary follows for $\theta_{f,0}$ from (54) combined with (50). As a result (55) reduces to result

$$\nabla_y \cdot (\lambda_f \nabla_y \theta_{f,0}) = 0. \quad (57)$$

Because of the scaling, the structure of equations corresponding to the matrix part of the problem take a slightly different form

Problem $P_{m,\epsilon}$: 0 order terms in ϵ

$$A_m \frac{\partial \theta_{m,0}}{\partial t} + \nabla_y \cdot [\rho(\theta_{m,0})\bar{v}_{m,0}] = 0 \quad (58)$$

$$\mu(\theta_{m,0})\bar{v}_{m,0} + K_m\left[\nabla_y p_{m,0} - \rho(\theta_{m,0})\bar{g}\right] = 0 \quad (59)$$

$$b(\theta_{m,0})\frac{\partial \theta_{m,0}}{\partial t} - \nabla_y \cdot (\lambda_m \nabla_y \theta_{m,0}) + c_m(\theta_{m,0})\bar{v}_{m,0} \cdot \nabla_y \theta_{m,0} = 0 \quad (60)$$

The equations that we have obtained are not yet "macroscopic". Intuitively, to describe the flow from the "macroscopic" point of view one has to "average" the relations between variables locally (i.e. over the cells). In this context we shall use the functions $\omega_j^C, j = 1, \ldots d$ and the tensor σ^C defined in the section (2) for $C = \Omega_0, C_{in} = \Omega_{m,0}, C_{out} = \Omega_{f,0}$. In what follows we drop the superscript ^{C}.

With ω_j one can observe that their linear combinations with coefficients $\frac{\partial \theta_{f,0}}{\partial x_j}$ satisfy the equation (55), and it remains valid if we add to the product of $(\omega_1, \omega_2, \ldots \omega_d) \cdot \nabla_x \theta_{f,0}$ an arbitrary function γ constant in y (but possibly dependent on x). Hence we set

$$\theta_{f,1} = \sum_{j=1}^{d} \omega_j \frac{\partial \theta_{f,0}}{\partial x_j} + \gamma(x)$$

and verify that it satisfies (57) with (51). When we take into account (as discussed above) that

$$\theta_{f,0} = \theta_{f,0}(x) \tag{61}$$

then we have

$$-\lambda_f \nabla_y \theta_{f,1} \cdot \eta = \lambda_f \nabla_x \theta_{f,0} \cdot \eta.$$

From (61), (40) we get

$$\nabla_y \cdot \bar{v}_{f,0} = 0 \tag{62}$$

and insert it into (53) differentiated with respect to y–variable which with (46) gives the system to be solved for $p_{f,1}$

$$\nabla_y \cdot [K_f \nabla_y p_{f,1}] = 0 \ , \quad y \in \Omega_{f,0} \tag{63}$$

$$K_f \nabla_y p_{f,1} \cdot \eta = -K_f \nabla_x p_{f,0} \cdot \eta \ , \quad s \in \Gamma_0 \tag{64}$$

In analogy with the treatment of $\theta_{f,1}$ we write

$$p_{f,1} = \sum_{j=1}^{d} \omega_j \frac{\partial p_{f,0}}{\partial x_j} + \xi(x) \tag{65}$$

As we see, functions from the family ω_j provide a (microstructure–dependent) skeleton over which we build functions periodic in the "fast" variable and describing dynamics (by gradient of $p_{f,0}$ or $\theta_{f,0}$) in the "slow" variable. To obtain the macroscopic description of the flow, we define

$$K_* = \frac{|\Omega_f^0|}{|\Omega_0|} K_f \left[\delta_{i,j} + \sigma_{i,j}\right] \tag{66}$$

and we integrate (53) over $\Omega_{f,0}$, divide the result by $|\Omega_0|$ (in order to retain characteristic proportions between cells and fissures) and get

$$\mu(\theta_{f,0}) \bar{V}_0 + K_* \left[\nabla_x p_{f,0} + \rho(\theta_{f,0}) \bar{g}\right] = 0 \tag{67}$$

with

$$\bar{V}_0 = \frac{1}{|\Omega_0|} \int_{\Omega_{f,0}} \bar{v}_{f,0}(y) dy$$

Equation (67) is a macroscopic form of Darcy's law decribing the relation between macroscopic seepage velocity $\bar{V}_0$ and pressure gradient $\nabla_x p_{f,0}$ quantified with macroscopic permeability coeffcient K_* (calculated with the help of microsctructure information.)

In order to get macroscopic relations corresponding to the continuity and energy equations, we integrate (41) over $\Omega_{f,0}$, scale the result with

$\frac{1}{|\Omega_0|}$ and get

$$\begin{aligned} \nabla_x \cdot \left(\rho_f(\theta_{f,0})\bar{V}_0\right) &+ A_f \frac{|\Omega_{f,0}|}{|\Omega_0|}\frac{\partial \theta_{f,0}}{\partial t} \\ +\frac{1}{|\Omega_0|}\int_{\Omega_{f,0}} (\nabla_y \cdot \rho(\theta_{f,0})\bar{v}_{f,1})\, dy &+ \frac{1}{|\Omega_0|}\int_{\Omega_{f,0}} (\nabla_y \cdot (\rho_1 \bar{v}_{f,0}))\, dy = 0 \end{aligned}$$

But from the divergence theorem, periodicity and (46) we have

$$\int_{\Omega_{f,0}} \nabla_y \cdot [\rho_1 \bar{v}_{f,0}] dy = \int_{\Gamma_0} \rho_1 \bar{v}_{f,0} \cdot \eta ds = 0, \tag{68}$$

so the last term in (68) vanishes. The calculation of the 3'rd term (with use of the divergence theorem) reveals the coupling with microstructiure via interface flux conditions, which leads to the memory effects exhibited by the flow (with η_m outer normal to blocks)

$$\begin{aligned} \int_{\Omega_{f,0}} \nabla_y \cdot \rho_f(\theta_{f,0}\bar{v}_{f,1}) dy &= \int_{\Gamma_0} \rho_{f,0}\bar{v}_{f,1} \cdot \eta ds = -\int_{\Gamma_0} \rho(\theta_{m,0})\bar{v}_{m,0} \cdot \eta_m ds \\ &= -\int_{\Omega_{m,0}} \nabla_y \cdot \rho(\theta_{m,0})\bar{v}_{m,0} dy \\ &= \int_{\Omega_{m,0}} A_m \frac{\partial \theta_{m,0}}{\partial t} dy. \end{aligned}$$

Finally, inserting the above formula to (68) and setting

$$A_* = A_f \frac{|\Omega_{f,0}|}{|\Omega_0|},$$

we get

$$A_* \frac{\partial \theta_{f,0}}{\partial t} + \nabla_x \left(\rho(\theta_{f,0}\bar{V}_0\right) = -\frac{1}{|\Omega_0|}\int_{\Omega_{m,0}} A_m \frac{\partial \theta_{m,0}}{\partial t} \tag{69}$$

which is the macroscopic continuity equation.

By a similar technique we obtain the macroscopic energy equation. We rewrite (56) using already known relations and get

$$\begin{aligned} b_f(\theta_{f,0})\frac{\partial \theta_{f,0}}{\partial t} &- \nabla_y \left(\lambda_f \nabla_y \theta_{f,2} + \lambda_f \nabla_x \theta_{f,1}\right) \\ &- \nabla_x \cdot (\lambda_f \nabla_x \theta_{f,0}) + \lambda_f \nabla_y \theta_{f,1}) \\ &+ c(\theta_{f,0})\bar{v}_{f,0} \cdot (\nabla_x \theta_{f,0} + \nabla_y \theta_{f,1}) = 0. \end{aligned}$$

Now we integrate as above and in order to calculate the integral of the last term we couple it with the solution of microscopic problem (as

before in order to get (46))

$$\frac{1}{|\Omega_0|}\int_{\Omega_{f,0}} c(\theta_{f,0})\bar{v}_{f,0}\cdot\nabla_y\theta_{f,1}dy$$
$$=\frac{c(\theta_{f,0})}{|\Omega_0|}\int_{\Omega_{f,0}}\left[\nabla_y\cdot(\theta_{f,1}\bar{v}_{f,0}dy-\theta_{f,1}\nabla_y\cdot\bar{v}_{f,0}\right]dy=0$$

Now with the second term equal to zero, the divergence theorem and periodicity yield

$$\int_{\Omega_{f,0}}\nabla_y\cdot(\theta_{f,1}\bar{v}_{f,0})dy=\int_{\Gamma_0}\theta_{f,1}\bar{v}_{f,0}\cdot\eta dy=0.$$

As a consequence we have

$$\frac{1}{|\Omega_0|}\int_{\Omega_{f,0}}\left\{b_f(\theta_{f,0})\frac{\partial\theta_{f,0}}{\partial t}-\nabla_y\left[\lambda_f\nabla_y\theta_{f,2}+\lambda_f\nabla_x\theta_{f,1}\right]\right\}dy$$
$$=\frac{1}{|\Omega_0|}\int_{\Omega_{m,0}}(b(\theta_{m,0}\frac{\partial\theta_{m,0}}{\partial t}+c(\theta_{m,0})\bar{v}_{m,0}\cdot\nabla_y\theta_{m,0})dy.$$

Finally, we set $b_*(\theta_{f,0})=b_f(\theta_{f,0})\frac{|\Omega_{f,0}|}{|\Omega_0|}$, define λ_* on the analogy with K_*, and get the macroscopic energy equation in the form

$$b_*(\theta_{f,0})\frac{\partial\theta_{f,0}}{\partial t} - \nabla_x\cdot(\lambda_*\nabla_x\theta_{f,0})c(\theta_{f,0})\bar{V}_0\cdot\nabla_x\theta_{f,0}$$
$$= \frac{1}{|\Omega_0}\int_{\Omega_{m,0}}\left(b(\theta_{m,0})\frac{\partial\theta_{m,0}}{\partial t}+c(\theta_{m,0})\bar{v}_{m,0}\cdot\nabla_y\theta_{m,0}\right)dy \quad (70)$$

In this way we have arrived at the complete model NFFM composed of macroscopic continuity (69), momentum (67), and energy (70) equations, microscopic conservation equations correspondingly (58), (59), (60) together with interface conditions (45), (43) complemented with definitions of coeffcients K_*, A_*, λ_*, b_* as above.

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