I. Introduction

1. Ordinary Differential Equations.

In most introductions to ordinary differential equations one learns a variety of methods for certain classes of equations, but the issues of existence and uniqueness are frequently ignored. The objective of such a theory is to recognize some useful sufficient conditions under which existence or uniqueness of a solution satisfying appropriate additional conditions are guaranteed. We begin with the initial value problem for an ordinary differential equation of the form

$$u'(t) = f(t, u(t)), \quad 0 \le t \le a. \tag{1}$$

Here we let $a > 0,\ b > 0$, and set $I_a = [0, a]$ and $\mathbb{B}_b(u_0) = \{s \in \mathbb{R} : |s - u_0| \le b\}$ for a given $u_0 \in \mathbb{R}$. Assume we are given a function $f : I_a \times \mathbb{B}_b(u_0) \to \mathbb{R}$. A *solution* on I_a is an absolutely continuous function $u : I_a \to \mathbb{R}$ with range $Rg(u) \subset \mathbb{B}_b(u_0)$ and which satisfies (1) at almost every $t \in I_a$. On a first reading one may replace 'absolutely continuous' with 'continuously differentiable' and 'measurable' by 'continuous' everywhere below, and the corresponding statements will follow from essentially the same proofs. Note then that one can use the continuity of the function on compact sets to obtain some of the boundedness assumptions.

Proposition 1. *Assume there is a function $K \in L^1(I_a)$ for which*

$$|f(t, x) - f(t, y)| \le K(t)|x - y|,\ t \in I_a,\ x, y \in \mathbb{B}_b(u_o). \tag{2}$$

Then any two solutions u_1, u_2 of (1) on I_a satisfy the estimate

$$|u_1(t) - u_2(t)| \le |u_1(0) - u_2(0)| e^{\int_0^t K(s)\,ds}, \quad t \in I_a. \tag{3}$$

Proof. Set $u = u_1 - u_2$. Then $u(t) = u(0) + \int_0^t u'$, so $|u(t)| \le |u(0)| + \int_0^t |u'|$. From (1) and (2) it follows that u satisfies

$$|u(t)| \le |u(0)| + \int_0^t K(s)|u(s)|\,ds, \quad t \in I_a.$$

The estimate (3) is a consequence of the following result with $w(t) = |u(t)|$.

Lemma 1. *Assume $k \in L^1(0, a),\ k \ge 0,\ g$ is absolutely continuous on $[0, a]$ and $w \in L^\infty(0, a)$ satisfies*

$$w(t) \le g(t) + \int_0^t k(s)w(s)\,ds, \quad 0 \le t \le a.$$

Then we have

$$w(t) \le g(0)e^{\int_0^t k(s)\,ds} + \int_0^t e^{\int_s^t k} g'(s)\,ds, \quad 0 \le t \le a.$$

Proof. Set $G(t) = \int_0^t k(s)w(s)\,ds$ and note that

$$G'(t) \le k(t)g(t) + k(t)G(t),$$

so we obtain

$$\begin{aligned}\frac{d}{dt}G(t)e^{-\int_0^t k} &\le -\frac{d}{dt}(e^{-\int_0^t k})g(t)\\ &= -\frac{d}{dt}(e^{-\int_0^t k}g(t)) + e^{-\int_0^t k}g'(t).\end{aligned}$$

Integrate this over $[0,t]$ to get

$$(G(t)+g(t))e^{-\int_0^t k} \le g(0) + \int_0^t e^{-\int_0^s k}g'(s)\,ds,$$

and then note that this yields the desired inequality. $\square$

This is an example of a *Gronwall inequality*. A variation is given next. Much more general versions are available, but these will suffice for our purposes.

Lemma 2. *Assume $k \in L^1(0,a)$, $k \ge 0$, $g \in L^\infty(0,a)$ and $w \in L^\infty(0,a)$ satisfies*

$$w(t) \le g(t) + \int_0^t k(s)w(s)\,ds, \quad 0 \le t \le a.$$

Then we have

$$w(t) \le g(t) + \int_0^t k(s)g(s)e^{\int_s^t k}\,ds\,.$$

Proof. In the proof above we had

$$\frac{d}{dt}G(t)e^{-\int_0^t k} \le e^{-\int_0^t k}k(t)g(t),$$

and from this we obtain by integrating (since $G(0)=0$)

$$G(t)e^{-\int_0^t k} \le \int_0^t e^{-\int_0^s k}k(s)g(s)\,ds,$$

and then multiplication by $e^{\int_0^t k}$ gives $w(t) \le g(t) + \int_0^t e^{\int_s^t k}k(s)g(s)\,ds$. $\square$

The assumption (2) is called a *Lipschitz condition* on the function; it essentially is a uniform bound on difference quotients and follows by the Mean Value theorem of calculus for continuously differentiable functions on bounded sets. It leads directly to *uniqueness* results like the following.

Corollary. *If $f(\cdot,\cdot)$ satisfies (2), there is at most one solution of the* **initial-value problem**

$$\begin{gathered}u'(t) = f(t,u(t)), \quad 0 \le t \le a,\\ u(0) = u_0\,,\end{gathered}$$

and it depends continuously on the initial value.

Proposition 2. *Assume there is a function $K \in L^1(I_a)$ for which*

$$|f(t,x)| \le K(t)(|x|+1),\ t \in I_a,\ x,y \in \mathbb{B}_b(u_o). \tag{3}$$

Then any solution u of (1) on I_a satisfies the estimates

$$|u(t)| + 1 \le (|u(0)|+1)e^{\int_0^t K(s)\,ds},$$
$$|u(t)| \le |u(0)| + (|u(0)|+1)\int_0^t K(s)e^{\int_s^t K}\,ds, \quad t \in I_a.$$

Exercise 1. Prove Proposition 2. Hint: Use Lemma 1 or Lemma 2 with $w(t) = |u(t)|+1$ and $g(t) = |u(0)|+1$ for the respective estimates. Show they are equivalent.

Proposition 3. *Assume there are functions $K_1,\ K_2,\ K_3 \in L^1(I_a)$ for which*

$$\begin{aligned} |f_1(t,x)| &\le K_1(t)(|x|+1),\\ |f_1(t,x) - f_2(t,x)| &\le K_2(t)|x|,\\ |f_2(t,x) - f_2(t,y)| &\le K_3(t)|x-y|,\ t \in I_a,\ x,y \in \mathbb{B}_b(u_o). \end{aligned} \tag{4}$$

Then any two solutions u_1, u_2 of the respective equations

$$u_1'(t) = f_1(t, u_1(t)), \quad u_2'(t) = f_2(t, u_2(t)),$$

on I_a satisfy the estimate

$$|u_1(t) - u_2(t)| \le g(t) + \int_0^t K_3(s)e^{\int_s^t K_3} g(s)\,ds \quad t \in I_a. \tag{5}$$

where $g(t) = |u_1(0) - u_2(0)| + \int_0^t K_2(s)(|u_1(s)+1)|\,ds$.

Proof. The difference $w(t) = u_1(t) - u_2(t)$ satisfies

$$|w(t)| \le |w(0)| + \int_0^t |w'(s)|ds \le |w(0)| + \int_0^t |f_1(s,u_1(s)) - f_2(s,u_2(s))|ds$$
$$\le |w(0)| + \int_0^t |f_1(s,u_1(s)) - f_2(s,u_1(s))|\,ds + \int_0^t |f_2(s,u_1(s)) - f_2(s,u_2(s))|\,ds$$
$$\le |w(0)| + \int_0^t K_2(s)|u_1(s)|\,ds + \int_0^t K_3(s)|w(s)|\,ds$$

and now apply Lemma 2. □

The assumption with K_2 shows that this measures the closeness of f_1 and f_2. Since the bound on f_1 shows that u_1 is bounded, we see that $g(t)$ is small if K_2 is small and the initial values are close. The estimate (5) shows that u_1 and u_2 are close when $g(t)$ is small. In summary, these mean that if the initial values are close and the functions f_1 and f_2 are close in the sense above, then the solutions u_1 and u_2 are close. The estimates make these assertions precise. In particular, they show that any solution of the initial-value problem for (1) is unique and it depends continuously on the initial condition u_0 and the function $f(\cdot,\cdot)$.

Exercise 2. In the situation of Proposition 2, show that f_2 satisfies a linear growth rate like the one on f_1.

Exercise 3. Assume the more general condition

$$(f(t,x)-f(t,y))(x-y)\le K(t)(x-y)^2 \tag{2'}$$

and obtain similar results.
Hint: Let $u_1(t)$ and $u_2(t)$ be solutions of (1) and set $w(t)=(u_1(t)-u_2(t))^2$. Then $w'(t)=2(u_1'(t)-u_2'(t))(u_1(t)-u_2(t))$.

Exercise 3'. Let $\sigma(s)=s^+=\begin{cases} s, & s\ge 0,\\ 0, & s<0.\end{cases}$ From (2'), show that $w(t)=\sigma(u_1(t)-u_2(t))$ satisfies $w'(t)\le K(t)w(t)$ and deduce that if $u_1(0)\le u_2(0)$ then $u_1(t)\le u_2(t)$ for $t\in I_a$.

Exercise 4. Show that there are many solutions of the initial value problem

$$u'=u^{\frac13},\quad t\ge 0,\qquad u(0)=0\,.$$

Show that $u(t)=0$ is the only solution of the problem

$$u'=-u^{\frac13},\quad t\ge 0,\qquad u(0)=0\,.$$

Hint: Use Exercise 3 with $K(t)=0$.

Now we turn to the issue of *existence* of solutions.

Theorem 1 (Cauchy-Picard). *In addition to the assumptions of Proposition 1, assume that $f(\cdot,x):I_a\to\mathbb{R}$ is measurable for each $x\in\mathbb{B}_b(u_0)$. Also, let $c>0$, $c\le a$ with $\int_0^c K\le\frac{b}{b+1}$ and assume $|f(t,u_0)|\le K(t)$, a.e. $t\in I_c$. Then there exists a (unique) solution of (1) on I_c with $u(0)=u_0$.*

Proof. First we note that for any measurable function $u:I_a\to\mathbb{B}_b(u_0)$, the composite function $f(\cdot,u(\cdot))$ is measurable on I_c, and from (2) we obtain

$$|f(t,u(t))|\le|f(t,u_0)|+|f(t,u(t))-f(t,u_0)|\le K(t)(1+b).$$

Define X to be the set of $u\in C(I_c,\mathbb{R})$ with each $u(t)\in\mathbb{B}_b(u_0)$. It follows that the function defined by

$$F[u](t)=u_0+\int_0^t f(s,u(s))\,ds,\quad t\in I_c,$$

maps X into itself and satisfies

$$|F[u](t)-F[v](t)|\le\int_0^t K(s)|u(s)-v(s)|\,ds,\quad t\in I_c,\quad u,v\in X.$$

It follows that

$$\|F[u]-F[v]\|_{C(I_c)}\le\frac{b}{1+b}\|u-v\|_{C(I_c)},$$

where we use the notation

$$\|u\|_{C(I_c)} = sup\{|u(t)| : \ t \in I_c)\},$$

and this shows that $F : X \to X$ is a strict contraction on the closed, convex set X in $C(I_c)$, hence, F has a unique fixed point,

$$(4) \qquad u \in X, \quad u(t) = u_0 + \int_0^t f(s, u(s))\, ds, \quad t \in I_c.$$

This u is the desired solution of the initial value problem.

The iteration scheme used to obtain the fixed point result used above provides a constructive method for solving the initial value problem. This can be numerically implemented, but there are much more efficient methods for computing approximate solutions. Theorem 1 is a *local* result, that is, it asserts the existence of a solution on some *possibly small* interval.

Exercise 5. Let $\alpha \geq 0$ and consider the initial value problem

$$u' = u^{1+\alpha}, \quad u(0) = u_0\,.$$

Find the intervals I_c for which there exists a (unique) solution on I_c. Specifically, show how c depends on u_0 and α.

Here is an example of a *global* result: it asserts the existence of a solution on an a-priori given interval.

Theorem 1'. *Let $K \in L^1(I_a)$, $u_0 \in \mathbb{R}$, and $f : I_a \times \mathbb{R} \to \mathbb{R}$ satisfy*

$$\begin{aligned}
&f(\cdot, x) \text{ is measurable for each } x \in \mathbb{R},\\
&|f(t,x) - f(t,y)| \leq K(t)|x-y|, \ x, y \in \mathbb{R}, \ a.e. \ t \in I_a,\\
&|f(t, u_o)| \leq K(t), \ a.e. \ t \in I_a.
\end{aligned}$$

Then there is a unique solution of (1) on I_a with $u(0) = u_0$.

Proof. As before the idea is to show that $F : X \to X$ has a unique fixed point. First show by a calculation that the iteration $F^N(\cdot)$ is a strict contraction on the closed, convex set X in $C(I_a)$, hence, $F^N(\cdot)$ has a unique fixed point,

$$u \in X, \quad F^N(u) = u\,.$$

But then we note that $F^N(F(u)) = F(u)$, so by uniqueness of the fixed point it follows that $F(u) = u$ as desired.

The preceding results suffice for most purposes, but we mention in passing some related results for the case in which the function $f(\cdot, \cdot)$ does not satisfy a Lipschitz estimate (2) as above. Here we consider the case in which this function is only *continuous* in the second argument. One says that a function which satisfies the first two conditions below, i.e., measurable in t and continuous in x, is of *Caratheodory type.*

Theorem 2 (Cauchy-Peano). *Let $K \in L^1(I_a)$, $u_0 \in \mathbb{R}$, and $f : I_a \times \mathbb{B}_b(u_0) \to \mathbb{R}$ satisfy*

$$\begin{aligned}
&f(\cdot, x) \text{ is measurable for each } x \in \mathbb{B}_b(u_0),\\
&f(t, \cdot) \text{ is continuous for a.e. } t \in I_a,\\
&|f(t,x)| \le K(t),\ x \in \mathbb{B}_b(u_0),\ \text{a.e. } t \in I_a,
\end{aligned}$$

and $\int_0^c K(s)\,ds \le b$. Then there exists a solution of (1) on I_c with $u(0) = u_0$.

Proof. As before, we first note that for any measurable function $v \in X$, the composition $f(\cdot, v(\cdot))$ is measurable, hence, integrable. Next, define for each integer $n \ge 1$ the function $\beta_n(t) = \max(0, t - \frac{1}{n})$. Each β_n is continuous and satisfies

$$|\beta_n(t) - \beta_n(s)| \le |t - s|, \tag{5.a}$$

$$|\beta_n(t) - t| \le \frac{1}{n}. \tag{5.b}$$

Let's show that there is for each $n \ge 1$ a solution $u_n \in C(I_c)$ of the equation

$$u_n(t) = u_0 + \int_0^{\beta_n(t)} f(s, u_n(s))\,ds, \quad t \in I_c. \tag{6}$$

On the interval $[0, \frac{1}{n}]$ we have $\beta_n(t) = 0$ so the solution there is identically equal to u_0. If the solution is known on $[0, \frac{j-1}{n}]$, then the values of $u_n(t)$ for $t \in [\frac{j-1}{n}, \frac{j}{n}]$ are determined recursively from the formula, due to the definition of β_n, and the solution of (6) is thereby extended to all of I_c.

The sequence $\{u_n\}$ satisfies the estimates

$$|u_n(t)| \le |u_0| + \int_0^c K(s)\,ds, \tag{7.a}$$

$$|u_n(t) - u_n(s)| \le \left|\int_{\beta_n(s)}^{\beta_n(t)} K(s)\,ds\right|. \tag{7.b.}$$

Since K is integrable, for each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\left|\int_s^t K(s)\,ds\right| < \varepsilon \text{ whenever } |t - s| < \delta,$$

and so from (7.b) and (5.a) we see that

$$|u_n(t) - u_n(s)| < \varepsilon \text{ whenever } |t - s| < \delta.$$

Thus, from (7.a) the sequence $\{u_n\}$ is *equibounded* and from the preceding it is *equicontinuous* on I_c. Ascoli's theorem asserts that from such a sequence we can extract a uniformly convergent subsequence, which we denote by $u_{n'}$, and it converges to a $u \in C(I_c)$. Finally, by writing

$$u_{n'}(t) = u_0 + \int_0^t f(s, u_{n'}(s))\,ds - \int_{\beta_{n'}(t)}^t f(s, u_{n'}(s))\,ds, \quad t \in I_c,$$

we can take the limit as $n' \to \infty$ to get (4) as before. Thus, the limit function u is a solution of (1) with $u(0) = u_0$ as desired.

Exercise 6. Explain why the iteration scheme of Theorem 1, $u_{n+1} = F[u_n]$, would not work in the proof of Theorem 2.

As before, the preceding gives existence of a local solution; the following is an example of a corresponding global existence result which applies to functions which are *linearly bounded* in the second argument.

Theorem 2'. *Let* $K \in L^1(I_a)$ *and* $f : I_a \times \mathbb{R} \to \mathbb{R}$ *satisfy*

$$\begin{aligned}
&f(\cdot, x) \text{ is measurable for each } x \in \mathbb{R}, \\
&f(t, \cdot) \text{ is continuous for a.e. } t \in I_a, \\
&|f(t,x)| \le K(t)(1 + |x|), \ x \in \mathbb{R}, \ \text{a.e. } t \in I_a.
\end{aligned}$$

Then for each $u_0 \in \mathbb{R}$ *there exists a solution of (1) on* I_a *with* $u(0) = u_0$.

Extensions. All of the results above are true also for *vector valued* functions, i.e., for solutions $u : I_a \to \mathbb{R}^N$ of (1) in which f is a function from $I_a \times \mathbb{R}^N$ into $\mathbb{R}^N$. Moreover, the results through Theorem 1' hold also with $\mathbb{R}$ replaced by any *Banach space*. The calculus of such vector valued functions is essentially the same as that for $\mathbb{R}$ valued functions; the Mean Value Theorem is the notable exception.

Finally, we consider the continuous dependence of the solution of the initial value problem on the data in the problem that is implicit in the function $f(\cdot, \cdot)$. A solution $u(t)$ of the problem is certainly continuous in t. Moreover, if $f(\cdot, \cdot)$ is continuous, it follows from the equation (1) that the solution is also continuously differentiable in t. Suppose that the function also depends continuously on a parameter $\alpha \in \mathbb{R}$, that is, $f_\alpha(t, u)$. Regard this parameter as another unknown, so we have the formally equivalent system

$$\begin{aligned}
u'(t) &= f_{v(t)}(t, u(t)), \quad u(a) = u_0\,, \\
v'(t) &= 0, \quad v(a) = \alpha\,.
\end{aligned}$$

for the unknown pair $u(t)\,, v(t)$, and the parameter now appears as initial data. From our estimate (3), it follows that $u(t)$ is continuous in α as well as u_0. Suppose now that the function $f_\alpha(\cdot, \cdot)$ depends analytically on the parameter α. Since the solution is obtained as the uniform limit of a sequence, and since each member of that sequence depends analytically on the parameter α, it follows that the solution itself depends analytically on the parameter α. These remarks hold as well for a family of parameters, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$.

2. First Order PDE.

The general partial differential equation of first order in two variables is an equation of the form

$$F(x, y, u, u_x, u_y) = 0 \tag{1}$$

for which a solution is a function $u = u(x, y)$ which satisfies (1) in the appropriate sense. Here we shall discuss the case of *quasilinear* equations, that is, equations which are linear in the highest order derivatives. For equations of first order as above, such an equation is of the form

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u) \tag{2}$$

determined by the three functions, $a(\cdot, \cdot, \cdot)$, $b(\cdot, \cdot, \cdot)$, $c(\cdot, \cdot, \cdot)$. We shall begin with an intuitive discussion of the solvability of (2). Our approach is geometric. The graph of a solution of (2) is a surface S in $\mathbb{R}^3$ given by $u = u(x, y)$. Let (x_0, y_0, u_0) be a point on S. The normal direction to the surface at (x_0, y_0, u_0) is given by $(p, q, -1)$, where $p = u_x(x_0, y_0, u_0)$, $q = u_y(x_0, y_0, u_0)$. The equation (2) requires only the single constraint,

$$a(x_0, y_0, u_0)p + b(x_0, y_0, u_0)q = c(x_0, y_0, u_0)\,, \tag{3}$$

so there is a one-parameter family of such directions. The direction (a, b, c) is tangent to *any* graph of a solution of (2) at (x_0, y_0, u_0), that is, it is a common tangent to all such solution surfaces. We define the *characteristic curves* to (2) to be those curves $(x(\tau), y(\tau), u(\tau))$ which are solutions of the system of ordinary differential equations

$$\begin{aligned} \frac{dx}{d\tau} &= a(x(\tau), y(\tau), u(\tau))\,, \\ \frac{dy}{d\tau} &= b(x(\tau), y(\tau), u(\tau))\,, \\ \frac{du}{d\tau} &= c(x(\tau), y(\tau), u(\tau))\,. \end{aligned} \tag{6}$$

It follows from the above that any such curve either remains in a solution surface or is completely disjoint from it. These observations suggest a technique for constructing the solution to an *initial value problem* for (2); this technique is called the *method of characteristics*.

Suppose we are given an *initial curve*

$$C : \begin{cases} x_0(s) \\ y_0(s) \\ u_0(s) \end{cases} .$$

The *initial value problem* for (2) and C is to find a solution u of (2) whose graph contains C, that is, a solution satisfying

$$u(x_0(s), y_0(s)) = u_0(s)\,. \tag{7}$$

We construct the desired solution as follows. Pick a point $(x_0(s), y_0(s), u_0(s))$ on the curve C. Using this as the initial value, solve the system (6) for a solution $(x(\cdot), y(\cdot), u(\cdot))$ satisfying the initial condition

$$(x(0), y(0), u(0)) = (x_0(s), y_0(s), u_0(s))\,.$$

This solution depends on the parameter s, so it consists of a triple of functions which we denote by

$$x = x(\tau, s), \quad y = y(\tau, s), \quad u = u(\tau, s)\,. \tag{8}$$

The system (8) will be the parametric representation of the desired surface, S, in which the curve C corresponds to $\tau = 0$. The solution u is recovered by resolving the first two equations in (8) for

$$\tau = \tau(x, y), \quad s = s(x, y)$$

and then substituting these into the third to obtain the desired solution in the form $u(x, y) = u(\tau(x, y), s(x, y))$, the function whose graph is S.

The success of the preceding technique depends on the relationship between the partial differential equation (2) and the initial curve C. We shall say that a curve in $\mathbb{R}^2$ is *characteristic* if it is the projection into $\mathbb{R}^2 \times \{0\}$ of a characteristic curve. Intuitively, we expect the following three cases to occur:

If $(x_0(s), y_0(s))$ is nowhere characteristic, then for each $u_0(\cdot)$ there will exist a unique solution to the initial value problem.

If C is a characteristic curve, then there will exist many solutions to the initial value problem. (One can be constructed as above from any nowhere characteristic curve which intersects C at a single point.)

If $(x_0(s), y_0(s))$ is characteristic and C is not a characteristic curve, then there will not exist any solution to the initial value problem. (In particular, the surface S will *not* be the graph of a function $u(\cdot, \cdot)$.)

Example 1. Consider the initial value problem

$$u_x + u_y = 1\,, \quad u(x, 0) = f(x)\,.$$

We can represent the initial curve by

$$C : \begin{cases} x_0(s) = s \\ y_0(s) = 0 \\ u_0(s) = f(s) \end{cases},$$

and the characteristic curves are given by the system

$$\frac{dx}{d\tau} = 1, \quad \frac{dy}{d\tau} = 1, \quad \frac{du}{d\tau} = 1\,.$$

The solutions of this system with initial values taken from the initial curve are just

$$x = \tau + s, \quad y = \tau, \quad u = \tau + f(s)\,,$$

and from these we calculate the unique solution

$$u = y + f(x - y)\,.$$

Example 2. With the same equation as in Example 1, we consider the curve

$$C : \begin{cases} x_0(s) = s \\ y_0(s) = s \\ u_0(s) = F(s) \end{cases}$$

for which the projection into $\mathbb{R}^2 \times \{0\}$ is characteristic. If there exists a solution u of the partial differential equation which contains C, then from (7) we obtain $F'(s) = u_0'(s) = 1$ as a *necessary* condition for existence. That is, $F(\cdot)$ must be chosen so that C is a characteristic curve. In that case, there are many such solutions of the initial value problem. One of these can be obtained from Example 1 by taking any function f for which $f(0) = F(0)$.

Example 3. Consider the initial value problem

$$xu_x + yu_y = u + 1\,, \quad u(x,x) = x^2\,.$$

The characteristic curves for this partial differential equation are determined by the system

$$\frac{dx}{d\tau} = x\,, \quad \frac{dy}{d\tau} = y\,, \quad \frac{du}{d\tau} = u + 1$$

and the initial condition is specified by the curve

$$x_0(s) = s\,, y_0(s) = s\,, u_0(s) = s^2\,.$$

Since the projection into $\mathbb{R}^2 \times \{0\}$ is characteristic, it is no surprise that the solution of this system, namely,

$$x = se^\tau \quad y = se^\tau \quad u = s^2 e^\tau + e^\tau - 1\,,$$

does not determine a function $u(x, y)$.

Example 4. Consider instead the same partial differential equation but with the initial condition $u(x, x^2) = x^2$. This can be written in parametric form as

$$x_0(s) = s\,, y_0(s) = s^2\,, u_0(s) = s^2\,,$$

and the curve on which the solution is specified is noncharacteristic, so we expect to get a unique solution. It is easy to see that this solution is

$$x = se^\tau \quad y = s^2 e^\tau \quad u = s^2 e^\tau + e^\tau - 1\,,$$

and we can solve this for the solution $u = y + \frac{x^2}{y} - 1$ of the initial value problem.

Theorem 3. *Assume the functions $a(x,y,u)$, $b(x,y,u)$, $c(x,y,u)$ are continuously differentiable in a domain of $\mathbb{R}^3$. Assume that the curve C : $(x_0(s), y_0(s), u_0(s))$ lies inside that domain and that it is continuously differentiable. Finally, assume that*

$$\begin{vmatrix} a(x_0(s), y_0(s), u_0(s)) & x_0'(s) \\ b(x_0(s), y_0(s), u_0(s)) & y_0'(s) \end{vmatrix} \neq 0\,. \tag{9}$$

Then there exists a unique continuously differentiable solution to the problem

$$(10) \qquad \begin{cases} a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u) \\ u(x_0(s), y_0(s)) = u_0(s) \end{cases}$$

in a neighborhood of C.

Proof. The proof consists of verifying that the hypotheses are sufficient to carry out the method of characteristics as outlined above. First, construct as above the solution of the system (6) starting from the curve C. The smoothness assumptions guarantee that the functions $x = x(\tau, s),\ y = y(\tau, s),\ u = u(\tau, s)$ are continuously differentiable solutions in a neighborhood of C. The assumption (9) shows that on C the Jacobian

$$\frac{\partial(x,y)}{\partial(\tau,s)} = \begin{vmatrix} \frac{\partial x}{\partial \tau} & \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial \tau} & \frac{\partial y}{\partial s} \end{vmatrix} = \begin{vmatrix} a(x_0(s), y_0(s), u_0(s)) & x_0'(s) \\ b(x_0(s), y_0(s), u_0(s)) & y_0'(s) \end{vmatrix}$$

is non-zero, so by continuity it is non-zero in a neighborhood of C. Then the implicit function theorem shows that the transformation $x = x(\tau, s),\ y = y(\tau, s)$ has a continuously differentiable inverse, so we obtain the continuously differentiable function $u(x, y)$ as above by substitution. From the chain rule we obtain

$$u_x = u_s s_x + u_\tau \tau_x\,, \quad u_y = u_s s_y + u_\tau \tau_y\,,$$

and this leads to

$$a u_x + b u_y = (a s_x + b s_y) u_s + (a\tau_x + b\tau_y) u_\tau\,.$$

From (6) we obtain

$$a s_x + b s_y = s_x x' + s_y y' = \frac{\partial}{\partial \tau} s(x(\tau), y(\tau)) = 0\,,$$
$$a \tau_x + b \tau_y = \tau_x x' + \tau_y y' = \frac{\partial}{\partial \tau} \tau(x(\tau), y(\tau)) = 1\,,$$

so we have

$$a u_x + b u_y = u_\tau = c\,.$$

Finally, note that

$$u(x_0(s), y_0(s)) = u(s, 0) = u_0(s)\,,$$

so the initial condition is also satisfied. This shows that u is a solution of (10) and establishes the existence of a solution.

Now, to show uniqueness, we let $u(x, y)$ be *any* solution of (10). Construct a curve on the graph of u by solving for each fixed s the pair of equations

$$\frac{dx}{d\tau} = a(x(\tau), y(\tau), u(x(\tau), y(\tau)))$$
$$\frac{dy}{d\tau} = b(x(\tau), y(\tau), u(x(\tau), y(\tau)))$$

with the initial conditions

$$(x(0), y(0)) = (x_0(s), y_0(s))\,.$$

Then define $u(\tau) = u(x(\tau), y(\tau))$ and note that

$$u'(\tau) = u_x x_\tau + u_y y_\tau = a u_x + b u_y = c\,,$$
$$u(0) = u(x(0), y(0)) = u(x_0(s), y_0(s)) = u_0(s)\,.$$

This shows that $(x(\tau), y(\tau), u(\tau))$ is a characteristic curve which which starts at $(x_0(s), y_0(s), u_0(s))$, so by the uniqueness of initial value problems for ordinary differential equations, the graph of u is specified by this information. $\square$

Exercises. Solve each of the following initial value problems explicitly.

1.
$$u_t + au_x = 0\,, \quad u(x,0) = f(x)\,, \qquad x \in \mathbb{R}\,,\ t > 0\,,$$
where a is a constant.

2.
$$xu_x + u_y = 1\,, \quad u(x,0) = e^x\,, \qquad x \in \mathbb{R}\,,\ t > 0\,.$$

3.
$$xu_x + u_y = u + 1\,, \quad u(x,0) = u_0(x)\,, \qquad x \in \mathbb{R}\,,\ t > 0\,.$$

4.
$$u_t + au_x = 0\,, \quad u(x,0) = f(x)\,, \quad u(0,t) = 0\,, \qquad x > 0\,,\ t > 0\,.$$

5.
$$u_t + au_x = 0\,, \quad u(x,0) = f(x)\,, \quad u(0,t) = u(1,t)\,, \qquad 0 < x < 1\,,\ t > 0\,.$$

6. Generalize the method of characteristics to equations of the form
$$\sum_{i=1}^{N} a_i(\mathbf{x}, u)\frac{\partial u}{\partial x_i} = c(\mathbf{x}, u)\,,$$
where $\mathbf{x} = (x_1, x_2, ..., x_N)$.

3. Second Order PDE.

We shall be concerned primarily with partial differential equations of the second order whose solutions are real-valued functions of (at least) two real variables. Letting $\mathbb{R}$ denote the set of real numbers and $\mathbb{R}^N$ the $N-dimensional$ Euclidean space, we can determine such an equation by a function $F: D \to \mathbb{R}$, where D is a subset of $\mathbb{R}^8$. The equation is given by

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0\,.$$

Equations as well as systems in one or more variables can be indicated in a similar manner. The order of the equation refers to the highest order of a derivative that appears in the equation. Finally, a solution of the partial differential equation above is a function $u: G \to \mathbb{R}$, where G is a subset of $\mathbb{R}^2$, which is twice differentiable with $(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) \in D$ for every $(x, y) \in G$ and such that the indicated equation is satisfied at each point of G.

Example. Let $f: G \to \mathbb{R}$ be continuous, where G is a disc in $\mathbb{R}^2$ with center (x_0, y_0). Then every solution of the equation

$$u_{xy} = f$$

can be given by the representation

$$u(x, y) = u(x_0, y) + u(x, y_0) - u(x_0, y_0) + \int_{x_0}^{x} \int_{y_0}^{y} f(s, t)\, dt\, ds\,.$$

Another representation for solutions of this equation is given by

$$\begin{aligned} u(x, y) = u(x_0 + y_0 - y, y) &+ \int_{x_0+y_0-y}^{x} u_x(s, x_0 + y_0 - s)\, ds \\ &+ \int_{x_0+y_0-y}^{x} \int_{x_0+y_0-s}^{y} f(s, t)\, dt\, ds\,, \end{aligned}$$

and there are many such representations. The first of these will give us the solution u if the values of u are known along the lines $x = x_0\,,\ y = y_0$. Likewise, the second representation determines u from known values of u and u_x along the line $x+y = x_0+y_0$. Thus each of these formulas or integral representations for solutions is appropriate for a different type of boundary value problem associated with the given equation.

A given partial differential equation may have many solutions. Our interest is in describing those solutions which satisfy the equation together with additional constraints, usually in the form of initial or boundary conditions. Specifically, we shall classify the equations of second order in such a way that the classification indicates what types of initial-boundary-value problems make *reasonable* or *well-posed* problems for that equation type. *Well-posed* refers to those problems for which there is exactly one solution, and it depends continuously on the data of the problem. We shall find representations for these solutions to be very useful in the discussion of initial- or boundary-value problems.

We begin with equations of the form

$$L[u] = f(x, y, u, u_x, u_y) \tag{1}$$

for which the *principle part* is given by

$$L[u] = a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy}\,, \tag{2}$$

where $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ and $c(\cdot,\cdot)$ are continuous functions on the open set G in $\mathbb{R}^2$. We shall assume that $a^2+b^2+c^2>0$ in G. The equation is called *degenerate* at a point in G where all the coefficients vanish; at such a point it is essentially an equation of (at most) first order.

An equation of the form (1) is called *semilinear* since the non-linearities in u and its derivatives occur only in those terms with at most first order derivatives; i.e., the principle part (2) is *linear* in u. If any one of the coefficients depends on u, u_x or u_y, then (1) is called *quasilinear*. Finally, if f is of the form

$$f = -A(x,y)u_x - B(x,y)u_y - C(x,y)u + F(x,y)\,,$$

then (1) is *linear*, for it is then of the form $T[u]=F$, where T is a linear operator between appropriate function spaces.

We investigate the effect on $L[\cdot]$ of a change of variable, $(x,y)\to(\sigma,\tau)$, given by

$$\sigma = \phi(x,y)\,, \quad \tau = \psi(x,y)\,, \qquad (x,y)\in G\,, \tag{3}$$

where ϕ and ψ are twice continuously differentiable functions for which the Jacobian, $\phi_x\psi_y - \phi_y\psi_x$, is different from 0 in G. It follows then that the map $(x,y)\to(\sigma,\tau)$ carries G onto an open set and is locally invertible. From a direct calculation by the chain rule we have

$$\begin{aligned}
u_x &= u_\sigma\sigma_x + u_\tau\tau_x\,,\\
u_y &= u_\sigma\sigma_y + u_\tau\tau_y\,,\\
u_{xx} &= u_{\sigma\sigma}\sigma_x^2 + 2u_{\sigma\tau}\sigma_x\tau_x + u_{\tau\tau}\tau_x^2 + \ \dots,\\
u_{xy} &= u_{\sigma\sigma}\sigma_x\sigma_y + u_{\sigma\tau}(\sigma_x\tau_y+\sigma_y\tau_x) + u_{\tau\tau}\tau_x\tau_y + \ \dots,\\
u_{yy} &= u_{\sigma\sigma}\sigma_y^2 + 2u_{\sigma\tau}\sigma_y\tau_y + u_{\tau\tau}\tau_y^2 + \ \dots,
\end{aligned} \tag{4}$$

where $+\ \dots$ denotes terms containing first order derivatives of u. Substitution of (4) into (1) gives the transformed equation

$$M[u] = g(\sigma,\tau,u,u_\sigma,u_\tau)$$

with the principle part

$$M[u] = A(\sigma,\tau)u_{\sigma\sigma} + 2B(\sigma,\tau)u_{\sigma\tau} + C(\sigma,\tau)u_{\tau\tau}\,,$$

whose coefficients are given by

$$\begin{aligned}
A &= a\sigma_x^2 + 2b\sigma_x\sigma_y + c\sigma_y^2\\
B &= a\sigma_x\tau_x + b(\sigma_x\tau_y+\sigma_y\tau_x) + c\sigma_y\tau_y\\
C &= a\tau_x^2 + 2b\tau_x\tau_y + c\tau_y^2\,.
\end{aligned} \tag{5}$$

Note that the system (5) may be written in the form

$$\begin{pmatrix} A & B\\ B & C\end{pmatrix} = \begin{pmatrix}\sigma_x & \sigma_y\\ \tau_x & \tau_y\end{pmatrix}\begin{pmatrix} a & b\\ b & c\end{pmatrix}\begin{pmatrix}\sigma_x & \tau_x\\ \sigma_y & \tau_y\end{pmatrix}.$$

Taking the determinant of both sides of this equation yields the identity

$$B^2 - AC = (\sigma_x\tau_y - \sigma_y\tau_x)^2(b^2-ac)\,, \tag{6}$$

and this shows that the *sign* of the quantity (b^2-ac) obtained from the principle part of (1) does not change under smooth transformations.

Definition. The quantity $b(x,y)^2 - a(x,y)c(x,y)$ is the *discriminant* of the equation (1) at the point $(x,y) \in G$.

The preceding discussion establishes the following result.

Theorem 4. *The sign of the discriminant at a point for the semilinear partial differential equation (1) of second order is invariant under a transformation (3) whose component functions are twice differentiable with non-zero Jacobian in a neighborhood of that point.*

Theorem 4 shows that the sign of the discriminant does not depend on the choice of coordinates in which the equation is expressed and, hence, that it provides a reasonable means of classifying (1) at a point in G. Even better reasons for using this classification will appear later.

Definition. The equation (1) is called *hyperbolic, parabolic* or *elliptic* at a point whenever the discriminant, $b^2 - ac$, is positive, zero, or negative, respectively, at that point. The equation is of a given type in the region G if it is of that type at every point of G. It is of *mixed type* on G if it has at least two types at (different) points of G.

Example. The semilinear equation

$$u_{xx} + xyu_{yy} + u_x^2 = 0$$

is hyperbolic in the two quadrants where $xy < 0$, it is parabolic on the axes, $xy = 0$, and it is elliptic where $xy > 0$. The equation is of mixed elliptic-parabolic type in the region where $xy \geq 0$.

Exercises.

1. Verify (6) directly from (5).
2. Show that L degenerates at (x_0, y_0) if and only if M degenerates at $s_0 = \phi(x_0, y_0)\,, t_0 = \psi(x_0, y_0)$.
3. Classify by type the following: Laplace's equation, $u_{xx} + u_{yy} = 0$, the diffusion equation, $u_t = u_{xx}$, and the wave equation, $u_{tt} - u_{xx} = 0$.
4. Show that the transformation $\sigma = x + y\,, \tau = x - y$ transforms the wave equation $u_{xx} - u_{yy} = 0$ to the form $u_{\sigma\tau} = 0$. Verify that every solution of the wave equation is of the form $u(x,y) = f_1(x+y) + f_2(x-y)$.
5. Discuss well-posedness of the problem $u_{xx} + u_{yy} = 0$ for $y > 0$ with $u(x,0) = f(x)$, $u_y(x,0) = g(x)$, $x \in \mathbb{R}$.

4. Characteristics and classification.

Suppose we are given the semi-linear second order equation (3.1). Our objective here is to find an appropriate pair of functions $\phi\,,\psi$ for which the change of variables (3.3) will reduce (1) to an equation with a simpler principle part. We shall assume that the coefficients in (3.1) are continuous and that $a(x,y)$ is non-zero in the region of our interest. Otherwise, we would either have $c(x,y) \neq 0$, in which case we do a construction as below with x and y interchanged, or else $c(x,y) = 0$, and then we can divide by the necessarily non-zero $b(x,y)$ to obtain a hyperbolic equation. (We assumed that (3.1) is non-degenerate, so not all of the coefficients can vanish.) From the implicit function theorem it follows that the function ϕ is a solution of the first order partial differential equation

$$a(x,y)\phi_x^2 + 2b(x,y)\phi_x\phi_y + c(x,y)\phi_y^2 = 0 \tag{1}$$

if and only if it is an integral of the ordinary differential equation

$$a(x,y)\left(\frac{dy}{dx}\right)^2 - 2b(x,y)\left(\frac{dy}{dx}\right) + c(x,y) = 0\,. \tag{2}$$

(Recall that an integral of (2) is a function ϕ whose level curves, $\phi(x,y) = c$, characterize solutions of (2) implicitly.) Hence, we seek " local" solutions, $y_1(x),\ y_2(x)$, respectively, of the pair of ordinary differential equations

$$\tag{3} \begin{aligned} \frac{dy}{dx} &= \frac{b(x,y) + (b(x,y)^2 - a(x,y)c(x,y))^{1/2}}{a(x,y)}\,, \\ \frac{dy}{dx} &= \frac{b(x,y) - (b(x,y)^2 - a(x,y)c(x,y))^{1/2}}{a(x,y)}\,, \end{aligned}$$

which is equivalent to (2). The number and type of these solutions depend on the discriminant.

Definition. An integral curve, $\phi(x,y) = c$, of (2) is a *characteristic curve*, and (2) is called the *characteristic equation* for the partial differential equation (3.1).

Example. The discriminant of the *Tricomi equation*

$$yu_{xx} + u_{yy} = 0$$

is given by $b^2 - ac = -y$, so the equation is hyperbolic where $y < 0$. The characteristic equation is $y(y')^2 + 1 = 0$. By a separation of variables, it follows that there are two characteristic curves through each point in the lower half-plane. These are given by $3x \pm 2(-y)^{3/2} = c$, and they degenerate to a single characteristic direction (the vertical) through each point of the x-axis where the equation is parabolic. There are no characteristics in the upper half-plane where the equation is elliptic.

We return to the equation (3.1). Suppose that (3.1) is hyperbolic in a region. Since the discriminant is positive, there are two distinct solutions, $y_1(x)$ and $y_2(x)$, of (2), and a corresponding pair of solutions of (1) is given by $\phi(x,y) = y_1(x) - y,\ \psi(x,y) = y_2(x) - y$. From the identity

$$\begin{vmatrix} \phi_x & \phi_y \\ \psi_x & \psi_y \end{vmatrix} = y_2' - y_1' = -\frac{2}{a}\sqrt{b^2 - ac}\,,$$

it follows that the transformation defined by (3.3) has a non-zero Jacobian in the region. Comparing (3.5) with (1), we find that the coefficients satisfy $A = C = 0$ in the region, and Theorem 4 then implies that $B^2 > 0$. Dividing the transformed equation by the non-zero coefficient B then yields

$$u_{\sigma\tau} = F(\sigma, \tau, u, u_\sigma, u_\tau)\,. \tag{4}$$

An additional change-of-variable, $\xi = \sigma + \tau, \eta = \sigma - \tau$ changes (4) to the form

$$u_{\xi\xi} - u_{\eta\eta} = G(\xi, \eta, u, u_\xi, u_\eta). \tag{5}$$

Hence, equation (3.1) can be reduced to either of the forms (4) or (5) in any region in which it is hyperbolic, and these equations are the canonical forms for hyperbolic semi-linear partial differential equations of second order.

If the equation (3.1) is parabolic in a region, then (3) reduces to a single equation, and there is only one characteristic curve through each point of the region. Let ψ be a solution of (1), and let ϕ be any smooth function for which the Jacobian, $\phi_x\psi_y - \phi_y\psi_x$, is non-zero. Then the coefficient C of the transformed equation is identically zero, and the parabolicity condition, $B^2 - AC = 0$, implies that $B = 0$. Dividing the transformed equation by the remaining necessarily non-zero coefficient A gives us

$$u_{\sigma\sigma} = F(\sigma, \tau, u, u_\sigma, u_\tau), \tag{6}$$

the canonical form for parabolic second-order semi-linear equations.

Example. The discriminant of the linear equation

$$xu_{xx} + 2xu_{xy} + |x|u_{yy} = 0$$

is 0 where $x \geq 0$ and $2x^2$ where $x < 0$. In the right half-plane the characteristic equation is $(y' - 1)^2 = 0$ with the solution $y = x + c$ and integral $\psi(x, y) = x - y$. Choose $\phi = x + y$, so the Jacobian is non-zero. The transformation $\sigma = x + y,\ \tau = x - y$ reduces the equation to the form $u_{\sigma\sigma} = 0$. Note that $u(\sigma, \tau) = f_1(\tau) + \sigma f_2(\tau)$ gives a family of solutions to the transformed equation, and the corresponding solutions of the original equation are given by $u(x, y) = f_1(x-y) + (x+y)f_2(x-y)$, where f_1 and f_2 are arbitrary twice differentiable functions.

In the half-plane where $x < 0$, the characteristic equation $(y')^2 - 2y' - 1$ has the solutions $y_1(x) = (1 + \sqrt{2})x + c,\ y_2(x) = (1 - \sqrt{2})x + c$ and the corresponding integrals $\phi = (1 + \sqrt{2})x - y,\ \psi = (1 - \sqrt{2})x - y$. The transformation (3.3) reduces the equation to the canonical form $u_{\sigma\tau} = 0$. If g_1 and g_2 are twice-differentiable functions, we obtain a solution $u(\sigma, \tau) = g_1(\sigma) + g_2(\tau)$ of the transformed equation and a corresponding solution of the original equation, $u(x, y) = g_1((1+\sqrt{2})x - y) + g_2((1 - \sqrt{2})x - y)$, in the left half-plane.

Consider finally the case in which the equation (3.1) is elliptic in a region. There are no real solutions of the characteristic equation (2), but we can apply the preceding technique if we assume further that the coefficients have analytic extensions to complex variables, x, y. Then we obtain a pair of complex-valued integrals, ϕ, ψ of (2) determined as above by a pair of complex-valued solutions of (3). The coefficients in (2) take real values for real x, y, so the same is true of y_1, y_2. But integrals are determined up to constants, so taking the conjugate of the identity $\phi_x + \phi_y y_1' = 0$ and noting that $\bar{y}_1' = y_2'$, we have $\psi = \bar{\phi} + c$ on the real plane. Hence, the transformation (3.3) introduces the complex variables $\sigma = \phi(x, y)$, $\tau = \psi(x, y)$ in which the equation (3.1) appears in the form (4). To obtain a form in real variables, introduce $\xi = Re(\sigma) = \frac{1}{2}(\sigma + \tau), \eta = Im(\sigma) = \frac{1}{2i}(\sigma - \tau)$ to obtain the canonical form for elliptic semi-linear equations,

$$u_{\xi\xi} + u_{\eta\eta} = G(\xi, \eta, u, u_\xi, u_\eta)\,. \tag{7}$$

The preceding discussion is summarized in the following.

Theorem 5. *Let the semi-linear partial differential equation (3.1) be given with twice continuously differentiable coefficients in a neighborhood of the point (x_0, y_0). If (3.1) is hyperbolic at (x_0, y_0), there is a transformation (3.3) whose component functions are twice continuously differentiable with non-zero Jacobian in a neighborhood of that point and which reduces (3.1) to the form (4). If (3.1) is parabolic at (x_0, y_0), there is a transformation as above which reduces (3.1) to the form (6). If (3.1) is elliptic at (x_0, y_0), and if its coefficientrs are analytic at (x_0, y_0), then there is a transformation as above which recuces (3.1) to the form (7).*

Proof. It suffices to show that the transformation (3.3) exists and is twice continuously differentiable at (x_0, y_0). This follows from the differentiability of the coefficients and the regularity of the solutions of the ordinary differential equations (2).

Example. The elliptic equation

$$u_{xx} + 2u_{xy} + 5u_{yy} = 0$$

has the characteristic equation $(y')^2 - 2y' + 5 = 0$ with complex-conjugate solutions $y_1(x) = (1 - 2i)x + c$, $y_2(x) = (1 + 2i)x + c$. We introduce the real variables $\sigma = Re(y - y_1(x)) = y - x$, $\tau = Im(y - y_1(x)) = 2x$ to obtain the canonical form $u_{\sigma\sigma} + u_{\tau\tau} = 0$.

Exercises. Reduce each of the following to a canonical form in regions where it is of a given type. Sketch the (real) characteristic curves in both the original and in the new coordinate planes.

1. $u_{xx} + 2u_{xy} + u_{yy} + u_x - u_y = 0$
2. $u_{xx} + 2u_{xy} + 5u_{yy} + 3u_x - u_y = 0$
3. $3u_{xx} + 10u_{xy} + 3u_{yy} = 0$
4. $yu_{xx} + u_{yy} = 0$
5. Show that a characteristic curve is invariant under the transformation (3.3).

5. Characteristics and Discontinuities.

Consider the situation of Section 3 in which we studied the effect of a change of variable (3.3) on the semilinear equation (3.1). This led to the new equation whose principle part has the coefficients A, B, C given by (3.5). Suppose now that $\phi(x,y)$ is a solution of

$$a\phi_x^2 + 2b\phi_x\phi_y + c\phi_y^2 = 0\,, \tag{1}$$

i.e., the curves $\phi(x,y) = k$ are characteristic. Then the coefficient A of u_{ss} in the transformed equation is identically zero, so the new equation takes the form

$$2Bu_{st} + Cu_{tt} = F(s,t,u,u_s,u_t)\,. \tag{2}$$

Note that (2) does not depend on the second order derivative u_{ss} in the direction normal to the characteristic curve $\phi(x,y) = k$, but it is an equation containing only directional derivatives of u and u_s in the direction determined by $t = \psi(x,y)$ with a non-zero component in the tangent direction. In particular, it may happen that some solution of (3.1) has a discontinuity in the second derivative in the direction *normal* to the characteristic curve.

Example. For any twice continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$, there is exactly one solution of

$$u_{xx} - u_{yy} = 0, \quad u(x,0) = f(x), \quad u_y(x,0) = 0\,.$$

It is given by $u(x,y) = \frac{1}{2}(f(x+y) + f(x-y))$. However, if $f(x) = x|x|$, then the second derivative $f''(x)$ does not exist. The function u given by our formula

$$u(x,y) = \begin{cases} x^2 + y^2, & \text{for } x \geq 0, \quad y \geq 0, \\ 2xy, & \text{for } |x| < y, \\ -(x^2+y^2), & \text{for } x \leq -y \leq 0 \end{cases}$$

is once continuously differentiable in the half-plane where $t \geq 0$ and is twice differentiable except on the lines $x = t,\ x = -t$. The second order derivatives are given by

$$u_{xx}(x,y) = u_{yy}(x,y) = \begin{cases} 2, & \text{for } x \geq 0, \quad y \geq 0, \\ 0, & \text{for } |x| < y, \\ -2, & \text{for } x \leq -y \leq 0. \end{cases}$$

In order to examine these derivatives more carefully, we introduce the characteristic coordinates $s = x + y,\ t = x - y$ to obtain the representation

$$u(s,t) = \begin{cases} \frac{1}{2}(s^2 + t^2), & \text{for } s \geq 0,\ t \geq 0, \\ \frac{1}{2}(s^2 - t^2), & \text{for } s \geq 0,\ t < 0, \\ -\frac{1}{2}(s^2 + t^2), & \text{for } s < 0,\ t < 0. \end{cases}$$

The second order derivatives are given by

$$u_{ss}(s,t) = \begin{cases} 1, & \text{for } s > 0,\ t > 0 \\ 1, & \text{for } s > 0,\ t < 0\,, \\ -1, & \text{for } s < 0,\ t < 0 \end{cases} \qquad u_{tt}(s,t) = \begin{cases} 1, & \text{for } s > 0,\ t > 0 \\ -1, & \text{for } s > 0,\ t < 0\,. \\ -1, & \text{for } s < 0,\ t < 0 \end{cases}$$

Along the line $s = 0$, the normal derivative u_{ss} exhibits a jump of magnitude 2, as does also the derivative u_{tt} along the line $t = 0$. Thus, the discontinuity in the initial data f at the origin is propogated along the characteristics $x \pm t = 0$.

This example illustrates that certain discontinuities in certain derivatives of the solution of (3.1) may exist along characteristic curves. We shall next show that this propogation of such discontinuities can occur *only* along *characteristic curves*. Namely, we consider discontinuities in the second order derivative in the direction *normal* to the curve. Let C be a curve given in the form $\phi(x, y) = k$ which separates the region into the two sets $\{(x, y)|\, \phi(x, y) < k\}$ and $\{(x, y)|\, \phi(x, y) > k\}$. Let $s = \phi(x, y)$ and choose t to be the parameter along C for which $t_x = \phi_y$, $t_y = -\phi_x$; that is, (s_x, s_y) is *orthogonal* to (t_x, t_y). At each point of C, define $[g]$ to be the magnitude of the jump of the piecewise continuous function g in the direction of increasing s. Let u be a solution of (3.1) for which u, u_x, u_y and each of their derivatives in the tangent direction along C are continuous, but for which the second order normal derivative u_{ss} takes a jump $[u_{ss}] \neq 0$. Consider the transformation (3.3), and assume it is twice continuously differentiable. From the chain rule we obtain the identities in (3.4) on either side of C. Use these to see that the jumps in the various second order derivatives along C satisfy the identities

$$\begin{aligned}
[u_{xx}] &= [u_{ss}]s_x^2 + 2[u_{st}]s_x t_x + [u_{tt}]t_x^2\,, \\
[u_{xy}] &= [u_{ss}]s_x s_y + [u_{st}](s_x t_y + s_y t_x) + [u_{tt}]t_x t_y\,, \\
[u_{yy}] &= [u_{ss}]s_y^2 + 2[u_{st}]s_y t_y + [u_{tt}]t_y^2\,.
\end{aligned} \tag{3}$$

Note that all the lower order terms in (3.4) were continuous, so their corresponding jumps were null, and the coefficients above are continuous, so they appear outside the brackets. Now from our assumptions on the smoothness of u along C, it follows that

$$[u_{st}] = [u_{tt}] = 0\,,$$

so we obtain

$$[u_{xx}] = [u_{ss}]\phi_x^2,\;\; [u_{xy}] = [u_{ss}]\phi_x\phi_y,\;\; [u_{yy}] = [u_{ss}]\phi_y^2. \tag{4}$$

Assume that the coefficients a, b, c and the function f in (3.1) are all continuous. Then we obtain from (3.1) the identity

$$a[u_{xx}] + 2b[u_{xy}] + c[u_{yy}] = 0\,,$$

and with (4) this gives (1). This shows that the curve $\phi(x, y) = k$ is necessarily a characteristic curve.

6. PDE in $\mathbb{R}^n$.

Consider the generalization of (3.1) to those semilinear second order partial differential equations which have solutions $u(\mathbf{x})$ which are functions of the n variables $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, and, hence, contain derivatives in n corresponding coordinate directions $\frac{\partial u}{\partial x_i} = u_{x_i}$, $i = 1, 2, \ldots, n$. We shall denote the variable coefficients by $\{a_{jl}(\mathbf{x})\}$, and the principle part is then given by

$$L[u] = \sum_{j=1}^{n} \sum_{l=1}^{n} a_{jl} u_{x_j x_l}, \quad x \in G \tag{1}$$

where G is an open set in $\mathbb{R}^n$. From the identity of the mixed derivatives, $u_{x_j x_l} = u_{x_l x_j}$, it follows that we may assume with no loss of generality that the coefficients in (1) are symmetric. Suppose now that $\mathbf{x}^0 \in G$ and make a change of variable

$$y_i = \sum_{j=1}^{n} c_{ij}(x_j - x_j^0)\,.$$

From the chain rule we obtain the identities

$$u_{x_j} = \sum_{i=1}^{n} u_{y_i} \frac{\partial y_i}{\partial x_j} = \sum_{i=1}^{n} c_{ij} u_{y_i}, \quad 1 \le j \le n\,,$$

$$u_{x_j x_l} = \sum_{k=1}^{n} \frac{\partial}{\partial y_k} u_{x_j} \frac{\partial y_k}{\partial x_l} = \sum_{k=1}^{n} \sum_{i=1}^{n} c_{ij} c_{kl} u_{y_i y_k}, \quad 1 \le j, l \le n\,.$$

Substitution of these quantities into (1) gives the transformed principle part

$$M[u] = \sum_{k=1}^{n} \sum_{i=1}^{n} A_{ki} u_{y_i y_k}$$

with coefficients

$$A_{ki} = \sum_{j=1}^{n} \sum_{l=1}^{n} c_{ij} c_{kl} a_{jl}, \quad 1 \le k, i \le n\,.$$

By introducing the matrices $\mathbf{a} = (a_{ij})$ and $\mathbf{c} = (c_{ij})$, the differential operators $\partial_x = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$ and $\partial_y = (\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n})$, and by denoting adjoints by a prime, we can express the above computations as follows:

$$L = \partial_x \mathbf{a} \partial_x' \quad \mathbf{y} = (\mathbf{x} - \mathbf{x}^0)\mathbf{c}'\,,$$
$$\partial_x = \partial_y \mathbf{c} \quad M = \partial_y(\mathbf{cac}')\partial_y'\,.$$

The coefficients in the transformed differential operator M are thus given by the matrix $\mathbf{A} = \mathbf{cac}'$. By the theory of symmetric quadratic forms, the matrix $\mathbf{c}$ can be chosen so as to obtain the matrix $\mathbf{A}$ at $\mathbf{y}_0 = \mathbf{0}$ in a form with diagonal consisting of $+1$, -1 or 0, and with only zeros in every position off of the diagonal. For any such choice of $\mathbf{c}$, the three numbers

$$n_1 = \text{number of } A_{jj} \text{ with } A_{jj} = 1,$$
$$n_{-1} = \text{number of } A_{jj} \text{ with } A_{jj} = -1,$$
$$n_0 = \text{number of } A_{jj} \text{ with } A_{jj} = 0,$$

remain *invariant*. Thus, these three numbers provide a means of classifying (1) independent of coordinates.

Definition. Let the second order differential operator L be given in a neighborhood of $\mathbf{x}^0 \in \mathbb{R}^n$, and let the numbers n_1, n_{-1}, n_0 be as given above. Then (1) is called *elliptic* at $\mathbf{x}^0$ if $n_1 = n$ or if $n_{-1} = n$, *parabolic* at $\mathbf{x}^0$ if $n_0 > 0, n_1 n_{-1} = 0$, *properly hyperbolic* at $\mathbf{x}^0$ if $n_0 = 0$ and either $n_1 = 1$ or if $n_{-1} = 1$, *ultra hyperbolic* if $n_0 = 0$, $n_1 > 1$ and $n_{-1} > 1$, and *hyper-parabolic* if $n_0 = 1$, $n_1 > 1$ and $n_{-1} > 1$.

Example. The equation

$$u_{x_1x_1} - 2u_{x_1x_3} + 2u_{x_2x_2} + 4u_{x_2x_3} + 6u_{x_3x_3} = 0$$

can be written in a symmetric matrix form with matrix

$$\mathbf{a} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 2 \\ -1 & 2 & 6 \end{pmatrix}.$$

The matrix

$$\mathbf{c} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

has the property that $\mathbf{cac}'$ is the identity matrix, hence, the transformation $\mathbf{y} = \mathbf{xc}$ reduces the above elliptic equation to the form

$$u_{y_1y_1} + u_{y_2y_2} + u_{y_3y_3} = 0$$

The preceding discussion shows that there is a change of variable which reduces (1) at the given point $\mathbf{x}^0$ to the form

$$M[u] = \sum_{j=1}^{n_1} u_{y_jy_j} - \sum_{j=n_1+1}^{n_2} u_{y_jy_j}$$

with $n_2 = n_1 + n_{-1}$ at the corresponding point $\mathbf{y}^0 = 0$. An attempt to find a transformation of the form $\mathbf{y} = \phi(\mathbf{x})$ which will accomplish this in a neighborhood of $\mathbf{x}^0$ leads to a system of partial differential equations in the components of ϕ which is overdetermined if $n \geq 3$, hence, an extension of Theorem 5 to higher dimensions is not possible, in general. However, if (1) has constant coefficients, the transformation $\mathbf{y} = \mathbf{xc}'$ introduces M with constant coefficients, and the indicated form above is attained at all points of $\mathbb{R}^n$. In particular, every linear equation with constant coefficients can be reduced to the form

$$\sum_{j=1}^{n} a_j u_{x_jx_j} + \sum_{j=1}^{n} b_j u_{x_j} + cu = f(\mathbf{x}) \tag{2}$$

where each of the coefficients a_j is ± 1 or 0.

Finally, we close with the following frequently useful observation. Suppose (2) has been arranged so that those coefficients a_j which vanish are those with $n - n_0 = n_2 < j \leq n$. Define the function

$$E(\mathbf{x}) = e^{\sum_{l=1}^{n_2} \frac{-b_l}{2a_l} x_l}$$

and make the change of variable $u(\mathbf{x}) = v(\mathbf{x})E(\mathbf{x})$. Then we obtain for $1 \leq j \leq n_1 + n_{-1}$

$$u_{x_j} = E(\mathbf{x})(v_{x_j} - \frac{b_j}{2a_j}v)\,,$$
$$u_{x_jx_j} = E(\mathbf{x})(v_{x_jx_j} - \frac{b_j}{a_j}v_{x_j} + \frac{b_j^2}{4a_j^2}v)\,.$$

Hence, from (2) we get

$$\sum_{j=1}^{n_1+n_{-1}} a_j u_{x_jx_j} + \sum_{j=n_1+n_{-1}+1}^{n} b_j u_{x_j} + (c - \sum_{j=1}^{n_1+n_{-1}} (\frac{b_j^2}{4a_j^2}))u = \frac{f(\mathbf{x})}{E(x)}\,. \tag{3}$$

Thus, every linear second order partial differential equation with constant coefficients can be reduced to the form (3). The point is that we can eliminate all of those first order terms b_j for which there is a corresponding non-zero entry a_j in the principle part. Specifically, *all* first order terms can be eliminated from a *non-parabolic* equation.

Exercises. Transform each of the following to the form (3).

1. $u_{xx} + u_{yy} + au_x + bu_y + cu = 0$.
2. $u_{xx} + u_{yy} + au_t + bu_x + cu_y = 0$.
3. $u_{xx} - u_{yy} + au_x + bu_y + cu = 0$.
4. $u_t + u_{xy} + au_x + bu_y + cu = 0$.
5. Give an example of a hyper-parabolic equation in $\mathbb{R}^3$.