## II. The Potential Equation

### 1. Introduction.

Our objective here is to show that the *Dirichlet boundary value problem* is well-posed for *Poisson's equation*

$$-\Delta u(x) = f(x) \tag{1}$$

which contains the Laplace operator $\Delta u \equiv \sum_{i=1}^{n} u_{x_i x_i}$. That is, in a domain $G$ in $\mathbb{R}^n$ we seek a solution of (1) which takes on prescribed values $u|_{\partial G} = g$ on the boundary $\partial G$. Thus, the data in the Dirichlet problem consists of the two functions, $f$ and $g$, and the domain $G$. It is this last condition, the domain $G$, which makes the problem difficult, since a solution which agrees with $g$ on the entire boundary $\partial G$ is demanded. In particular, it is rather easy to find solutions of the partial differential equation itself, but the real problem is to satisfy the boundary conditions. Although *local* solutions of (1) might seem interesting, or even a single global solution on all of $G$, they usually represent very little progress towards solving the boundary value problem.

In Section 2 we shall derive a representation for solutions of (1) in terms of $f$ on $G$ and the function $u$ and its normal derivative on $\partial G$. Various properties of solutions of *Laplace's equation*

$$-\Delta u(x) = 0 \tag{2}$$

will be derived in Section 3 from this representation. This representation will be refined in Section 4 to construct a candidate for the solution of the Dirichlet problem, and we shall use this to solve explicitly the Dirichlet problem for very special domains. After examining some of the consequences of solvability on the sphere, we turn finally to prove existence of a solution on general domains.

### 2. A Fundamental Representation.

Let $G$ denote a normal domain in $\mathbb{R}^n$. A *harmonic function* on $G$ is a function $u \in C^2(G)$ which satisfies *Laplace's equation*

$$\Delta u = 0 \tag{1}$$

in $G$. As is often the case with linear equations, certain special solutions which depend only on the distance from some point can be useful in the construction of other solutions. In particular, solutions of (1) of the form $u(x) = w(r)$, where $r = \|x - \xi\| = (\sum_{i=1}^{n}(x_i - \xi_i)^2)^{\frac{1}{2}}$ is the distance from $x$ to $\xi$, are given by

$$u(x) = \begin{cases} ln(\frac{1}{r})\,, & n = 2\,, \\ \frac{1}{r^{n-2}}\,, & n \geq 3\,. \end{cases}$$

Each of these functions has a singularity at $\xi$ characterized by the dimension $n$. It will be convenient to define the *singular solution* of (1) by

$$s(x,\xi) \equiv \begin{cases} \frac{1}{2\pi} ln(\frac{1}{r})\,, & n = 2\,, \\ \frac{1}{(n-2)\omega_n r^{n-2}}\,, & n \geq 3\,, \end{cases} r = \|x - \xi\|\,,$$

where $\omega_n$ denotes the surface area of the unit sphere in $\mathbb{R}^n$. In particular, $\omega_2 = 2\pi$ and $\omega_3 = 4\pi$. The coefficients are chosen to *normalize* the singularity for calculations to follow below.

Recall that the *Divergence Theorem*

$$\int_G \nabla \cdot \vec{F}\, dx = \int_{\partial G} \vec{F} \cdot \vec{\nu}\, dS$$

holds on $G$ when $\vec{F} = (F_1, F_2, \ldots, F_n)$ has components in $C(\bar{G}) \cap C^1(G)$ and $\vec{\nu}$ is the unit outward normal on the boundary, $\partial G$. If $u$, $v \in C^1(\bar{G}) \cap C^2(G)$, then we can set $F_j = u\, v_{x_j}$ in the divergence theorem to obtain the *First Green's Identity*

$$\int_G (u\Delta v + \nabla u \cdot \nabla v)\, dx = \int_{\partial G} u\, \frac{\partial v}{\partial \nu}\, dS \tag{2}$$

in which $\nabla v = (v_{x_1}, v_{x_2}, \ldots, v_{x_n})$ is the *gradient* of $v$ and

$$\frac{\partial v}{\partial \nu} = \nabla v \cdot \vec{\nu}$$

is the *normal derivative*, the directional derivative in the normal direction, $\nu$. A corresponding result holds with $u$ and $v$ interchanged, and by subtracting these we obtain the *Second Green's Identity*

$$\int_G (u\Delta v - v\Delta u)\, dx = \int_{\partial G} (u\, \frac{\partial v}{\partial \nu} - v\, \frac{\partial u}{\partial \nu})\, dS\,. \tag{3}$$

From (2) we obtain our first *uniqueness* result for Laplace's equation. If $u = v$ is a harmonic function in $G$ and belongs to $C^1(\bar{G})$, then from (2) follows

$$\int_G (\|\nabla u\|^2)\, dx = \int_{\partial G} u\, \frac{\partial u}{\partial \nu}\, dS\,,$$

so if additionally $u = 0$ on $\partial G$, it follows that $u$ is constant on $G$, hence, identically zero. This shows that there is at most one such solution in $C^1(\bar{G}) \cap C^2(G)$ of the *boundary value problem* for *Poisson's equation*

$$\begin{aligned} -\Delta u(x) &= f(x), \quad x \in G\,, \\ u(x) &= g(x), \quad x \in \partial G\,. \end{aligned} \tag{4}$$

We shall show later that there is at most one solution of (4) in the larger class $C(\bar{G}) \cap C^2(G)$. Note also that by setting $v = 1$ in (3) we obtain *Gauss' law*

$$\int_G \Delta u\, dx = \int_{\partial G} \frac{\partial u}{\partial \nu}\, dS\,. \tag{5}$$

This is related to the conservation of mass in many applicatons, and it will be useful in some computations below.

We want now to obtain a representation of a solution of the boundary value problem (4) in terms of integrals over $G$ and around $\partial G$. The identity (3) is our starting point. We shall apply it to a smooth function $u$ and choose $v(x) = s(x, \xi)$, a singular solution of Laplace's equation (1) with singularity at $\xi \in G$. Because of the singularity we cannot apply (3) directly to the region $G$, but we shall instead apply it to the region $G_\varepsilon$ obtained from $G$ by deleting the sphere $S_\varepsilon$ of radius $\varepsilon > 0$ centered at $\xi$. Now since $\Delta s(\cdot, \xi) = 0$ in $G_\varepsilon$, $\partial G_\varepsilon = \partial G - \partial S_\varepsilon$, and at $x \in \partial S_\varepsilon$ the unit outward normal to $\partial G_\varepsilon$ is given by $-\frac{x-\xi}{\varepsilon}$, we obtain for $n \geq 3$

$$
\begin{aligned}
-\int_{G_\varepsilon} \Delta u(x) s(x,\xi)\, dx &= \int_{\partial G} \big(u(x)\frac{\partial s(x,\xi)}{\partial \nu} - s(x,\xi)\frac{\partial u(x)}{\partial \nu}\big)\, dS_x \\
&\quad - \int_{\partial S_\varepsilon} \big(u(x)\big(-\frac{1}{\omega_n \varepsilon^{n-1}}\big) - \frac{1}{(n-2)\omega_n \varepsilon^{n-2}}\frac{\partial u(x)}{\partial \nu}\big)\, dS_x\,. \qquad (6)
\end{aligned}
$$

We consider the behaviour of each term as $\varepsilon \to 0$. From (5) we get the estimate

$$
\Big|\int_{\partial S_\varepsilon} \frac{\partial u(x)}{\partial \nu}\, dS_x\Big| = \Big|\int_{S_\varepsilon} \Delta u(x)\, dx\Big| \leq \big(\frac{\omega_n \varepsilon^n}{n}\big) \max_{x \in S_\varepsilon} |\Delta u(x)|
$$

in which the coefficient on the last term is the volume of the sphere $S_\varepsilon$. This gives the first limit

$$
\lim_{\varepsilon \to 0}\Big\{\frac{1}{\varepsilon^{n-2}}\int_{\partial S_\varepsilon} \frac{\partial u(x)}{\partial \nu}\, dS_x\Big\} = 0\,.
$$

Since the surface area of $S_\varepsilon$ is given by $\omega_n \varepsilon^{n-1}$, we obtain

$$
\begin{aligned}
\Big|\int_{\partial S_\varepsilon} u(x)\, dS_x - \omega_n \varepsilon^{n-1} u(\xi)\Big| &= \Big|\int_{\partial S_\varepsilon} (u(x) - u(\xi))\, dS_x\Big| \\
&\leq \omega_n \varepsilon^{n-1} \max_{x \in \partial S_\varepsilon} |u(x) - u(\xi)|\,.
\end{aligned}
$$

The continuity of the function $u$ then yields the limit

$$
\lim_{\varepsilon \to 0}\Big\{\frac{1}{\omega_n \varepsilon^{n-1}}\int_{\partial S_\varepsilon} u(x)\, dS_x\Big\} = u(\xi)\,.
$$

Thus, we have shown that the right side of (6) converges as $\varepsilon \to 0$, and so it follows that the left side does also, and we have obtained the *integral representation*

$$
u(\xi) = \int_{\partial G} \big(s(x,\xi)\frac{\partial u(x)}{\partial \nu} - u(x)\frac{\partial s(x,\xi)}{\partial \nu}\big)\, dS_x - \int_G \Delta u(x) s(x,\xi)\, dx\,. \qquad (7)
$$

This identity expresses the value of a function $u \in C^1(\bar{G}) \cap C^2(G)$ at a point $\xi \in G$ in terms of $\Delta u$ in the interior $G$ and both of $u$ and $\frac{\partial u}{\partial \nu}$ on the boundary $\partial G$. Thus, the first term in (7) containing the normal derivative of $u$ on the boundary is a defect in the representation of a solution of the boundary value problem (4). However the identity (7) will be very useful in the following discussion of properties of harmonic functions, and we shall return in Section 4 to make the appropriate modifications to eliminate the defect in the representation, i.e., to eliminate the term involving $\frac{\partial u}{\partial \nu}$.

**Exercises.**

1. If $u$ is a function whose value at $x$ depends only on the distance $r = \|x - \xi\|$ from some point $\xi$, show that $\Delta u = r^{1-n}(r^{n-1}u_r)_r$. Then show that such a function is harmonic if and only if it is of the form $c_1 + c_2 s(x, \xi)$, where $c_1$ and $c_2$ are constants and $s(x, \xi)$ is the singularity function.

2.a If $\omega_n$ denotes the surface area of the unit sphere in $\mathbb{R}^n$, then the surface area of the sphere of radius $\varepsilon$ is $\omega_n \varepsilon^{n-1}$, and its volume is given by

$$v_n(\varepsilon) = \int_0^\varepsilon \omega_n \varepsilon^{n-1}\, d\varepsilon = \frac{\omega_n \varepsilon^n}{n}.$$

2.b Show that

$$v_{n+1}(1) = 2\int_0^1 v_n(\sqrt{1-x^2})\, dx\,.$$

Note that $v_2(1) = \pi = \frac{\omega_2}{2}$, so $\omega_2 = 2\pi$. Also $v_3(1) = \frac{4\pi}{3}$, so $\omega_3 = 4\pi$. Show that $\omega_4 = 2\pi^2$.

2.c Note that

$$v_{n+1}(1) = 2\int_0^1 \frac{\omega_n}{n}(1-x^2)^{\frac{n}{2}}\, dx = 2\frac{\omega_n}{n}\int_0^{\frac{\pi}{2}} \cos^{n+1}(\theta)\, d\theta\,,$$

hence,

$$\omega_{n+1} = 2\omega_n \frac{n+1}{n}\int_0^{\frac{\pi}{2}} \cos^{n+1}(\theta)\, d\theta\,.$$

Derive the equality

$$\int_0^{\frac{\pi}{2}} \cos^{n+1}(\theta)\, d\theta = \frac{n}{n+1}\int_0^{\frac{\pi}{2}} \cos^{n-1}(\theta)\, d\theta$$

from which we get the recursive formula

$$\omega_{n+1} = \frac{n-2}{n-1}\frac{\omega_n \omega_{n-1}}{\omega_{n-2}} \quad n \geq 3$$

2.d Show by induction that

$$\omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\,, \quad n \geq 1,$$

where $\Gamma(\cdot)$ is the Gamma function.

3. Show that if $u$ is harmonic in $G$ and $u \in C^1(\bar{G})$, then

$$\int_{\partial G} \frac{\partial u}{\partial \nu}\, dS_x = 0\,.$$

4.a State and prove a uniquenes result for the *Neumann problem* which asks for a solution of

$$-\Delta u(x) = f(x), \quad x \in G\,,$$
$$\frac{\partial u}{\partial \nu} = g(x), \quad x \in \partial G\,.$$

In particular, show that any solution is determined up to a constant and that this is the best that can be done.

4.b Show that a necessary condition for the Neumann problem to have a solution is that the compatibility condition

$$\int_G f(x)\,dx + \int_{\partial G} g(x)\,dS_x = 0$$

hold.

4.c Discuss uniqueness for solutions of the *Robin problem*

$$\begin{aligned} -\Delta u(x) &= f(x), \quad x \in G\,, \\ \alpha u(x) + \beta \frac{\partial u}{\partial \nu} &= g(x), \quad x \in \partial G\,, \end{aligned}$$

in which $\alpha^2 + \beta^2 > 0$ and $\alpha\beta \geq 0$.

5. Verify the formula (7) for the case $n = 2$.

6. The boundary value problem (4) in $\mathbb{R}^1$ can be stated as

$$\begin{aligned} -u''(x) &= f(x), \quad 0 < x < 1, \\ u(0) &= g_1, \quad u(1) = g_2, \end{aligned}$$

where $G = (0, 1)$. Show that for any function $f$ which is continuous on $[0, 1]$ the unique solution is given by

$$u(x) = (1 - x)g_1 + xg_2 + \int_0^1 G(x, s)f(s)\,ds$$

where

$$G(x, s) = \begin{cases} (1 - x)s, & 0 \leq s < x \leq 1, \\ (1 - s)x, & 0 \leq x < s \leq 1. \end{cases}$$

## 3. Harmonic Functions.

Here we shall develop some of the properties of harmonic functions such as differentiability, mean value theorems on spheres, and maximum principle. All of these results are consequences of the representation (2.7), and it leads to the very useful notion of *subharmonic* functions. Throughout this section, $G$ is a bounded open set in $\mathbb{R}^n$ and $S(\xi, r)$ denotes the sphere of radius $r > 0$ and center at $\xi \in \mathbb{R}^n$.

**Proposition 1.** *If $u$ is harmonic in $G$ then $u$ is infinitely differentiable in $G$.*

*Proof.* Choose $S(\xi, r)$ so that $\bar{S}(\xi, r) \subset G$. Applying the representation (2.7) to $u$ on $S(\xi, r)$ gives

$$u(y) = \int_{\partial S(\xi,r)} \big(s(x, y)\frac{\partial u(x)}{\partial \nu} - u(x)\frac{\partial s(x, y)}{\partial \nu}\big)\,dS_x\,. \tag{1}$$

Since the integrand has continuous derivatives of all orders in the variable $y$ in some neighborhood of $\partial S(\xi, r) \times \{\xi\}$ in $\mathbb{R}^{2n}$, it follows by Leibinitz rule that $u$ is infinitely differentiable at $\xi$.

One of the characterizations of harmonic functions is that they have the *mean value property.* We show the necessity of this property here, and in Section 4 we shall obtain the converse.

**Proposition 2: Mean Value Theorem.** *If $u$ is harmonic in the sphere $S(\xi, R)$ and continuous on $\bar{S}(\xi, R)$, then $u(\xi)$ is equal to the mean value of $u$ on $\partial S(\xi, R)$*

*Proof.* Let $0 < r < R$ and set $y = \xi$ in the identity above. Since $s(x, \xi) = \frac{r^{2-n}}{\omega_n(n-2)}$ is constant, Gauss' law (2.5) shows that the integral of the first term is zero. Also, $\frac{\partial}{\partial\nu} = \frac{\partial}{\partial r}$, so we obtain

$$u(\xi) = \frac{1}{\omega_n r^{n-1}} \int_{\partial S(\xi,r)} u(x)\, dS_x \tag{2}$$

from an explicit calculation. Expressing this integral in the polar angle gives

$$u(\xi) = \frac{1}{\omega_n} \int_{\partial S(\xi,1)} u(ry)\, d\Omega_y \,,$$

and the uniform continuity of $u$ on $S(\xi, R)$ permits us to take the limit $r \to R$ in the above to obtain the desired result, namely, (2) with $r = R$.

**Corollary 1.** *The value of $u$ at $\xi$ is equal to the mean value of $u$ in $S(\xi, R)$.*

*Proof.* For each $r$, $0 < r < R$, we have from (2)

$$\omega_n r^{n-1} u(\xi) = \int_{\partial S(\xi,r)} u(x)\, dS_x \,.$$

Integration of this identity gives

$$\frac{\omega_n R^n}{n} u(\xi) = \int_0^R \int_{\partial S(\xi,r)} u(x)\, dS_x\, dr = \int_{S(\xi,R)} u(x)\, dx \,.$$

The coefficient of $u(\xi)$ is the volume of $S(\xi, R)$, so this is the desired result.

Many of the following results will depend only on the mean value properties of harmonic functions. Moreover, we will find it useful to consider the following class of functions.

**Definition.** A function $u \in C(\bar{G})$ is called *subharmonic* in $G$ if for every sphere $S(\xi, r)$ with $\bar{S}(\xi, r) \subset G$ we have

$$u(\xi) \le \frac{1}{\omega_n r^{n-1}} \int_{\partial S(\xi,r)} u(x)\, dS_x \,. \tag{3}$$

Of course it follows as before that any such function necessarily satisfies the *volume* form of the sub-mean-value property, namely,

$$u(\xi) \le \frac{1}{\frac{\omega_n r^n}{n}} \int_{S(\xi,r)} u(x)\, dx \tag{4}$$

for any sphere $S(\xi, r)$ as above.

We are motivated by the proof of Proposition 2 to use the identity (2.7) on a general function $u \in C^2(G)$ restricted to the sphere $S(\xi, r)$ with $\bar{S}(\xi, r) \subset G$. By evaluating it at the center of the sphere and using Gauss' theorem on the first term as before, we obtain

$$u(\xi) = \frac{1}{\omega_n r^{n-1}} \int_{\partial S(\xi,r)} u(x)\, dS_x + \int_{S(\xi,r)} (-\Delta u(x))(s(x,\xi) - \frac{r^{2-n}}{\omega_n(n-2)})\, dx \,.$$

Since $s(x, \xi) - \frac{1}{\omega_n r^{n-2}} \ge 0$ for $x \in S(\xi, r)$, we obtain the following extension of the Mean Value Theorem.

**Proposition 2': Sub Mean Value Theorem.** *If $u \in C^2(G) \cap C(\bar{G})$ satisfies*

$$\Delta u(x) \geq 0\,, \;\; x \in G$$

*then $u(\xi)$ is subharmonic in $G$.*

The Mean Value Theorem provides one proof of the following fundamental result. Another is outlined in the exercises.

**Proposition 3: Maximum Principle.** *Let $u$ be subharmonic in $G$. Then the maximum of $u$ on $\bar{G}$ is attained on $\partial G$.*

*Proof.* Since $u$ is continuous on the compact set $\bar{G}$, there is a $y \in \bar{G}$ such that $u(x) \leq u(y)$ for all $x \in \bar{G}$. Assume that $y \in G$, and let $R > 0$ be the distance from $y$ to $\partial G$. Since $u$ is subharmonic, we have

$$\omega_n R^{n-1} u(y) \leq \int_{\partial S(y,R)} u(x)\, dS_x \leq \int_{\partial S(y,R)} u(y)\, dS_x = \omega_n R^{n-1} u(y)\,,$$

so we obtain

$$\int_{\partial S(y,R)} [u(y) - u(x)]\, dS_x = 0\,.$$

But the integrand is continuous and non-negative, so it must be identically zero. Hence, by the choice of $R$, the maximum value is attained somewhere on $\partial G$.

**Corollary 2: Strong Maximum Principle.** *Let $u$ be subharmonic in $G$. If $G$ is connected then either $u$ is constant or*

$$u(x) < \max_{y \in \bar{G}} u(y)\,, \quad x \in G.$$

*Proof.* Consider the set $A = \{x \in G : u(x) = \max_{y \in \bar{G}} u(y)\}$. Since $u$ is continuous, this set is closed in $G$, that is, its complement is open in $G$. From the proof of the Maximum Principle, if $x \in A$ then there is a sphere $S(x,R) \subset A$ with $R > 0$, so $A$ is also open in $G$. Since $G$ is connected, either $A = G$ or $A = \emptyset$.

**Corollary 3: Order and Uniqueness.** *Let $G$ be open and bounded in $\mathbb{R}^n$, $f_1$ and $f_2 \in C(G)$, $g_1$ and $g_2 \in C(\partial G)$. Let $u_1\,, u_2 \in C^2(G) \cap C(\bar{G})$ be corresponding solutions of the boundary value problem (2.4). If $f_1(x) \leq f_2(x)$, $x \in G$ and $g_1(x) \leq g_2(x)$, $x \in \partial G$, then $u_1(x) \leq u_2(x)$, $x \in G$. In particular, there is at most one solution $u \in C^2(G) \cap C(\bar{G})$ of the boundary value problem (4).*

*Proof.* It suffices to note that $u \equiv u_1 - u_2$ is subharmonic on $G$ and non-negative on $\partial G$.

A similar argument shows that a solution of the Dirichlet problem (4) depends continuously on the boundary values: if $f_1 = f_2$, then

$$\max_{x \in \bar{G}} |u_1(x) - u_2(x)| = \max_{x \in \partial G} |g_1(x) - g_2(x)|\,.$$

**Exercises.**

1. Verify the statement in the proof of Theorem 1 concerning the differentiability of the integrand.

2. Provide the details in the following proof of the Maximum Principle. Let $y \in G$ be a point at which the maximum of $u$ occurs. If $\varepsilon > 0$ is sufficiently small, then $v(x) \equiv u(x) + \varepsilon|x - y|^2$ is larger at $y$ than at any $x \in \partial G$. Thus, $v$ has a maximum at some $z \in G$. But $\Delta v(x) = \Delta u(x) + 2\varepsilon n > 0$ for $x \in G$, while $\frac{\partial^2 v(z)}{\partial x_j^2} \leq 0$ for all $j$, $1 \leq j \leq n$.

3. Let $u_1,\ u_2$ be subharmonic in $G$. If $c_1, c_2 \geq 0$, show that $c_1u_1 + c_2u_2$ and $\max\{u_1, u_2\}$ are subharmonic in $G$.

**4. Green's Function.**

Our objective here is to obtain an integral representation and existence theorem for the solution of the Dirichlet problem for Laplace's equation (2.1). We showed already in the previous section that there is at most one such solution and that it depends continuously on the boundary values, so it will follow that the problem is well-posed.

Recall the discussion which led to the representation (2.7) in terms of the singularity function. If $u \in C^1(\bar{G}) \cap C^2(G)$, then we showed that

$$(1) \qquad u(\xi) = \int_{\partial G} \big(s(x,\xi)\frac{\partial u(x)}{\partial \nu} - u(x)\frac{\partial s(x,\xi)}{\partial \nu}\big)\, dS_x - \int_G \Delta u(x) s(x,\xi)\, dx\,.$$

Were it not for the term containing the normal derivative of $u$, (1) would provide a representation of a smooth solution of our boundary value problem (2.4) in terms of the data, that is, in terms of the boundary values of $u$, and $\Delta u(x)$ in the interior, hence, (1) would provide a means of defining a likely candidate for the solution. This motivates the following construction.

**Definition.** The Green's function for the region $G$ is given by

$$G(x,\xi) = s(x,\xi) - w(x,\xi)$$

where $s(x,\xi)$ is the singular solution of Laplace's equation and for each $\xi \in G$, the function $w(\cdot,\xi)$ is a harmonic function in $C^1(\bar{G})$ for which

$$w(x,\xi) = s(x,\xi)\,, \quad x \in \partial G.$$

It follows from the uniqueness result above that there is at most one such function, $w(\cdot,\cdot)$, so any Green's function is uniquely determined by the region $G$. By repeating the argument that led to (2.7) but with the Green's function $G(\cdot,\cdot)$ in place of the singular solution $s(\cdot,\cdot)$, we obtain

$$(2) \qquad u(\xi) = -\int_{\partial G} u(x)\frac{\partial G(x,\xi)}{\partial \nu}\, dS_x - \int_G \Delta u(x) G(x,\xi)\, dx$$

for any normal domain $G$ and $u \in C^1(\bar{G}) \cap C^2(G)$. Note that by using the Green's function we have eliminated the troublesome term involving $\frac{\partial u}{\partial \nu}$ on the boundary. In particular, if $u \in C^1(\bar{G})$ is harmonic in $G$, then

$$(3) \qquad u(\xi) = -\int_{\partial G} u(x)\frac{\partial G(x,\xi)}{\partial \nu}\, dS_x\,.$$

Fix $\xi$, $x \in G$, and then define the functions

$$v(y) \equiv G(y,\xi)\,,\ w(y) \equiv G(y,x) \quad y \in G.$$

Then $v$ and $w$ are harmonic in $G - \{\xi\}$ and $G - \{x\}$, respectively, and $v = w = 0$ on $\partial G$. By applying Green's identity (3) to these functions on the domain $G_\varepsilon \equiv G - S(\xi,\varepsilon) - S(x,\varepsilon)$ we obtain

$$\int_{\partial S(\xi,\varepsilon)} \frac{\partial v}{\partial \nu} w - \frac{\partial w}{\partial \nu} v\, dS_y + \int_{\partial S(x,\varepsilon)} \frac{\partial v}{\partial \nu} w - \frac{\partial w}{\partial \nu} v\, dS_y = 0\,.$$

By calculations as those leading to (2.7), we find that the above converges to

$$-w(\xi) + v(x) = 0\,,$$

so we have established the following.

**Proposition 1: Symmetry of Green's Function.** *For all $\xi$, $x \in G$, we have*

$$G(\xi,x) = G(x,\xi)\,.$$

In the remainder of this section, we construct the Green's function for two important examples.

*The Half Space.* Consider the half space defined by $G = \mathbb{R}^n_+ = \{x \in \mathbb{R}^n : \ x_n > 0\}$. Let $\tilde{x} \equiv (x_1, x_2, \ldots, -x_n)$ denote the point *symmetric* to $x \equiv (x_1, x_2, \ldots, x_n) \in G = \mathbb{R}^n_+$, and define then

$$G(x,\xi) = s(x,\xi) - s(x,\tilde{\xi}).$$

It is easy to check that $G(\xi,x)$ is the Green's function for $G$. Furthermore, note that the two terms in $G(\cdot,\cdot)$ are symmetric with respect to $\partial G$, and it is this symmetry that gives the required equality on the boundary. Then we compute

$$\frac{\partial G}{\partial \nu} = -\frac{\partial G}{\partial x_n} = \frac{-2\xi_n}{\omega_n |x-\xi|^n}\,, \quad x_n = 0\,,\ \xi_n > 0\,,$$

so from (3) we obtain

$$u(\xi) = \frac{2}{\omega_n} \int_{\mathbb{R}^{n-1}} \frac{u(x_1, x_2, \ldots, x_{n-1}, 0)\xi_n\, dx_1 \ldots dx_{n-1}}{[(x_1-\xi_1)^2 + \ldots (x_{n-1}-\xi_{n-1})^2 + \xi^2]^{\frac{n}{2}}} \tag{4}$$

for *bounded* harmonic functions on $G$.

**Proposition 2.** *Assume $g \in C(\bar{\mathbb{R}}^{n-1})$ and define*

$$u(\xi) = \frac{2}{\omega_n} \int_{\mathbb{R}^{n-1}} \frac{g(x_1, x_2, \ldots, x_{n-1})\xi_n\, dx_1 \ldots dx_{n-1}}{[(x_1-\xi_1)^2 + \ldots (x_{n-1}-\xi_{n-1})^2 + \xi^2]^{\frac{n}{2}}}\,. \tag{4}$$

*Then $u$ is harmonic (and, hence, $C^\infty$) in $G$ and*

$$\lim_{\substack{x \to x_0 \\ x \in G}} u(x) = g(x_0), \quad x_0 \in \mathbb{R}^{n-1}\,.$$

The proof follows by a direct calculation; see the next case below.

*Remark.* The preceding construction depends on the symmetric placement of the points $\xi$ and $\tilde{\xi}$ with respect to the boundary, $\partial G$. In the case of a quadrant $G \equiv \{(x_1, x_2) : \ x_1 > 0, \ x_2 > 0\}$ in the plane, we start with a point $(\xi_1, \xi_2) \in G$ and then reflect about $x_1 = 0$ to get $(-\xi_1, \xi_2)$ and then reflect both of these about $x_2 = 0$ to get $(-\xi_1, -\xi_2)$ and $(\xi_1, -\xi_2)$ for which the four points are symmetric about both $x_1 = 0$ and $x_2 = 0$, hence, about the boundary of $G$. The Green's function is then given by

$$G(x, \xi) = s(x, \xi) - s(x, (-\xi_1, \xi_2)) - s(x, (\xi_1, -\xi_2)) + s(x, (-\xi_1, -\xi_2)) \,.$$

Note that the alternating signs of symmetric points cause the corresponding terms to add to zero on $\partial G$. A similar construction in $\mathbb{R}^3$ leads to 8 such terms. These are examples of the classical *method of images.*

*The Sphere.* Now we consider the sphere $S(0, R)$. Again the idea is to construct the Green's function by taking advantage of some symmetry of the region. For each point $\xi \in S(0, R)$ we define the corresponding *symmetric* point $\tilde{\xi}$ to be that point outside of $S(0, R)$ which lies on the same ray through the origin with the requirement that $\|\xi\| \|\tilde{\xi}\| = R^2$. Thus, we have $\tilde{\xi} = \frac{R^2 \xi}{\|\xi\|^2}$. We shall justify this definition. For any point $x \in \mathbb{R}^n$, we have

$$\begin{aligned} \|x - \tilde{\xi}\|^2 &= \|x\|^2 - 2x \cdot \tilde{\xi} + \|\tilde{\xi}\|^2 \\ &= \|x\|^2 - 2x \cdot \xi \frac{R^2}{\|\xi\|^2} + \frac{R^4}{\|\xi\|^2} \,, \end{aligned}$$

and if in addition $\|x\| = R$, it follows that

$$\begin{aligned} \|x - \tilde{\xi}\|^2 &= \frac{R^2}{\|\xi\|^2} (\|\xi\|^2 - 2x \cdot \xi + \|x\|^2) \\ &= \frac{R^2}{\|\xi\|^2} \|\xi - x\|^2 \,. \end{aligned}$$

Thus, for each $x \in \partial S(0, R)$ we have

$$\|x - \xi\| = \|x - \tilde{\xi}\| \frac{\|\xi\|}{R} \,.$$

This shows that the ratio of the distances to $\xi$ and to $\tilde{\xi}$ is *constant* from each boundary point of the sphere, and this is the symmetry condition that we needed. By placing a singularity function at $\tilde{\xi}$ and scaling it with $\frac{\|\xi\|}{R}$, we obtain the desired Green's function. Thus, we define

$$G(x, \xi) = s(x, \xi) - s(\frac{\|\xi\|}{R} x, \frac{\|\xi\|}{R} \tilde{\xi}) \,.$$

It is immediate from our calculations above that this is the Green's function for the sphere.

Now we calculate the normal derivative $\frac{\partial G}{\partial \nu}$ on $\partial S(0,R)$. In order to use the chain rule we note that $G(x,\xi)$ is a difference of a function of $r = \|x-\xi\|$ and a function of $\tilde{r} = \frac{\|\xi\|}{R}\|x-\tilde{\xi}\|$. As before, we compute

$$\frac{\partial r}{\partial x_i} = \frac{x_i - \xi_i}{r}\,, \quad \frac{\partial s(x,\xi)}{\partial x_i} = \frac{-1}{\omega_n r^{n-1}} \frac{x_i - \xi_i}{r}\,,$$

and similarly we obtain

$$\frac{\partial \tilde{r}}{\partial x_i} = \frac{\|\xi\|^2}{R^2} \frac{x_i - \tilde{\xi}_i}{\tilde{r}}\,, \quad \frac{\partial s(\frac{\|\xi\|}{R}x, \frac{\|\xi\|}{R}\tilde{\xi})}{\partial x_i} = \frac{-1}{\omega_n \tilde{r}^{n-1}} \frac{\frac{\|\xi\|^2}{R^2}(x_i - \tilde{\xi}_i)}{\tilde{r}} = \frac{-1}{\omega_n \tilde{r}^n}\left(\frac{\|\xi\|^2}{R^2}x_i - \xi_i\right).$$

Finally, on $\partial S(0,R)$ we have $r = \tilde{r}$ and $\nu = \frac{1}{R}x$, so we have from above

$$\frac{\partial G(x,\xi)}{\partial \nu} = \frac{-1}{\omega_n r^n} \sum_{i=1}^{n} \frac{1}{R} x_i[(x_i - \xi_i) - (\frac{\|\xi\|^2}{R^2}x_i - \xi_i)] = \frac{-(R^2 - \|\xi\|^2)}{\omega_n R \|x-\xi\|^n}\,.$$

Substituting this into (3), we obtain

$$u(\xi) = \int_{\partial S(0,R)} \frac{(R^2 - \|\xi\|^2) u(x)}{\omega_n R \|x-\xi\|^n}\, dS_x\,.$$

We have shown that if $u$ is harmonic in the sphere $S(0,R)$, continuously differentiable on $\bar{S}(0,R)$, and if $u$ satisfies the Dirichlet boundary condition in (4), then

$$u(\xi) = \int_{\partial S(0,R)} \frac{(R^2 - \|\xi\|^2) g(x)}{\omega_n R \|x-\xi\|^n}\, dS_x\,, \quad \xi \in S(0,R)\,. \tag{5}$$

Note that for $\xi = 0$ this is just the Mean Value Theorem. Also, by using a uniform continuity argument like that in the proof of the Mean Value Theorem, we establish (5) for those $u$ which are harmonic in the sphere $S(0,R)$ but only continuous on the closure $\bar{S}(0,R)$. In fact the formula (3) can be extended likewise for those regions $G$ which can be uniformly approximated from within. Finally, we note the special case of (5), namely,

$$1 = \int_{\partial S(0,R)} \frac{(R^2 - \|\xi\|^2)}{\omega_n R \|x-\xi\|^n}\, dS_x\,, \quad \xi \in S(0,R)\,, \tag{6}$$

which follows for the function $u(x) = 1$. This result will be useful in a computation below.

The representation (5) suggests the following major result of this chapter.

**Proposition 3: Poisson's Representation.** *Let the function $g$ be continuous on the sphere $\partial S(0,R)$ and define a function $u : \bar{S}(0,R) \to \mathbb{R}$ by (5) for $\xi \in S(0,R)$ and $u(\xi) = g(\xi)$ for $\xi \in \partial S(0,R)$. Then $u$ is harmonic and, hence, infinitely differentiable in $S(0,R)$ and continuous on $\bar{S}(0,R)$.*

*Proof.* First we show that $u$ is harmonic in $S(0,R)$. Let $\xi \in S(0,R)$. The denominator in (5) is infinitely differentiable and bounded away from 0, so it follows that

$u$ is infinitely differentiable and we may compute derivatives in the integral. In particular, we have

$$\Delta u(\xi) = \int_{\partial S(0,R)} g(x) \Delta \frac{(R^2 - \|\xi\|^2)}{\omega_n R \|x - \xi\|^n} \, dS_x \, .$$

It suffices then to show that the integrand is harmonic in $\xi$. For this, we need only to compute the derivative in the integrand directly. Alternatively, we can note that $G(x, \xi)$ is harmonic in $\xi$, and by the equality of mixed derivatives the same holds for each $\frac{\partial G(x,\xi)}{\partial x_i}$ and, hence, for $\frac{\partial G(x,\xi)}{\partial \nu}$.

We need only to verify that $u$ is continuous at each point of $\partial S(0, R)$. Let $\varepsilon > 0$ and $x_0 \in \partial S(0, R)$. Since $g$ is continuous at $x_0$ there is a $\delta > 0$ such that $\|x - x_0\| < \delta$ and $x \in \partial S(0, R)$ imply that $\|g(x) - g(x_0)\| < \varepsilon$. Let $C_1$ denote those $x \in \partial S(0, R)$ at which $\|x - x_0\| < \delta$ and let $C_2 = \partial S(0, R) - C_1$. From (5) and (6) it follows that for any $\xi \in S(0, R)$ we have

$$u(\xi) - g(x_0) = \int_{\partial S(0,R)} \frac{(R^2 - \|\xi\|^2)(g(x) - g(x_0))}{\omega_n R \|x - \xi\|^n} \, dS_x \, .$$

This integral can be expressed as the sum of corresponding integrals over $C_1$ and $C_2$ to give the estimate

$$|u(\xi) - g(x_0)| \le \varepsilon \frac{(R^2 - \|\xi\|^2)}{\omega_n R} \int_{C_1} \frac{dS_x}{\|x - \xi\|^n} + \frac{(R^2 - \|\xi\|^2)}{\omega_n R} 2M \int_{C_2} \frac{dS_x}{\|x - \xi\|^n}$$

where $M$ is the maximum of $g$ on $\partial S(0, R)$. The first term is at most $\varepsilon$ by (6). If we restrict $\xi$ so that $\|\xi - x_0\| < \frac{\delta}{2}$, then for all $x \in C_2$ we have $\|x - \xi\| \ge \frac{\delta}{2}$ and, hence, the second term above is bounded by $\frac{R^2 - \|\xi\|^2}{\omega_n R} 2M (\frac{2}{\delta})^n \omega_n R^{n-1}$. By requiring that $\|\xi - x_0\|$ be sufficiently small, we force $R^2 - \|\xi\|^2$ to be small and, hence, the latter term can be made as small as desired. That is, we can make $|u(\xi) - g(x_0)| < 2\varepsilon$, and this finishes the proof.

**Exercises.** 1. Show that (3) holds if $G$ is convex and if $u$ is harmonic in $G$ and continuous on $\bar{G}$.

2. Show that the Green's function is positive in $G$ and that the kernel in (3) is non-negative.

3. Show that (5) is equivalent to

$$u(\xi) = \int_{\partial S(0,R)} \frac{R^{n-2}(R^2 - \|\xi\|^2) g(Rx)}{\omega_n \|x - \xi\|^n} \, d\Omega_x \, , \quad \xi \in S(0, R) \, ,$$

where $\Omega$ is the polar angle.

4. Show directly that $\frac{(R^2 - \|\xi\|^2)}{\|x - \xi\|^n}$ is harmonic in $\xi$.

5. Prove Proposition 2.

6. Set $S = S(0, R)$ and $\partial_+ S = \{x \in \partial S : x_n > 0\}$. Assume $g_+ \in C(\overline{\partial_+ S})$ and let $g$ be the zero extension of $g_+$ to all of $\partial S$. Find the boundary values of the function $u$ defined by (5).

**5. Consequences of Poisson's Formula.**

In addition to establishing the existence of a solution to the Dirichlet problem on the sphere, Proposition 4.3 has certain other implications. Some of these are of independent interest, and others will be used later in our discussion of the Dirichlet problem on more general regions. The first result is a converse to the Mean Value Theorem.

**Proposition 1.** *If $u$ is continuous on the open set $G$ and satisfies the Mean Value Property at every $\xi \in G$, namely, there is an $\varepsilon > 0$ such that*

$$u(\xi) = \frac{1}{\omega_n r^{n-1}} \int_{\partial S(\xi,r)} u(x)\, dS_x$$

*for every sphere $S(\xi, r)$ for which $0 < r < \varepsilon$, then $u$ is harmonic in $G$.*

*Proof.* For each sphere $S(\xi, r)$ there is an harmonic function $v$ on $S(\xi, r)$ which is continuous on $\bar{S}(\xi, r)$ and equals $u$ on $\partial S(\xi, r)$. The function $u - v$ has the mean value property in $S(\xi, r)$ and so the Maximum principle (Proposition 3.3) shows that $u-v$ and $v-u$ have their maximal values on $\partial S(\xi, r)$. Hence, $u = v$ in $S(\xi, r)$, and this shows that $u$ is harmonic in every sphere within $G$.

In combination with the Mean Value theorem, this shows that a continuous function on an open set $G$ is harmonic in $G$ if and only if it has the mean value property at every point in $G$. Another consequence of Proposition 1 is the following.

**Reflection Principle.** *Suppose $G$ is an open set which lies on one side of a hyperplane and whose boundary intersects this hyperplane in a subset $\Gamma$. Let $u$ be harmonic in $G$, continuous on $G \cup \Gamma$, and $u = 0$ on $\Gamma$. For each point $x \in \mathbb{R}^n$ denote by $\tilde{x}$ the point symmetric to $x$ with respect to the hyperplane. Define $u$ on $\tilde{G} \equiv \{\tilde{x} : x \in G\}$ by $u(\tilde{x}) = -u(x),\ x \in G$. Then $u$ is harmonic in the interior of $G \cup \Gamma \cup \tilde{G}$.*

Suppose now that we have a sequence of functions $\{u_k\}$ which are harmonic in the open set $G$ and continuous on the closure, $\bar{G}$. If the sequence converges uniformly on the boundary, $\partial G$, then it follows from the Maximum Principle that this sequence converges uniformly on all of $\bar{G}$ to a function $u$ which is necessarily continuous on $\bar{G}$. Let $y$ be a point in $G$ and choose $0 < r < R$ such that the closure of the sphere $S(y, R)$ is contained in $G$. By Poisson's representation (5) we have for every $k \geq 1$

$$u_k(\xi) = \int_{\partial S(y,R)} \frac{(R^2 - \|\xi\|^2) u_k(x)}{\omega_n R \|x - \xi\|^n}\, dS_x\,, \quad \xi \in S(y, r)\,,$$

and the denominator is bounded by $(R - r)^{-n}$, uniformly for $x \in \partial S(y, R)$ and $\xi \in S(y, r)$. From the uniform convergence of the sequence if follows that we may take the limit in the above identity to obtain

$$u(\xi) = \int_{\partial S(y,R)} \frac{(R^2 - \|\xi\|^2) u(x)}{\omega_n R \|x - \xi\|^n}\, dS_x\,, \quad \xi \in S(y, r)\,.$$

But $r < R$ is arbitrary, so this last identity holds for all $\xi \in S(y, R)$, hence, $u$ is harmonic in $S(y, R)$. Furthermore, if $D$ denotes any derivative (of any order), then we have also

$$Du_k(\xi) = \int_{\partial S(y,R)} D\Big(\frac{(R^2 - \|\xi\|^2)}{\omega_n R \|x - \xi\|^n}\Big) u_k(x)\, dS_x\,,$$

and the integrand is uniformly bounded as before for $\xi \in S(y,r)$ and $x \in \partial S(y,R)$. Hence, the sequence $\{Du_k\}$ converges uniformly on $S(y,r)$. Note that every compact subset of $G$ can be covered by a finite number of spheres, each of which is properly contained in a sphere whose closure is in $G$. These remarks prove the following.

**Proposition 2: Weierstrass.** *Let the sequence $\{u_k\}$ of functions harmonic in the bounded open set $G$ and continuous on $\bar{G}$ be uniformly convergent on $\partial G$. Then the sequence converges uniformly on $\bar{G}$ to a function $u$ which is harmonic in $G$ and continuous on $\bar{G}$. Furthermore, if $D$ denotes any derivative, then the sequence $\{Du_k\}$ converges uniformly on every compact set in $G$ to $Du$.*

Now suppose that $u$ is harmonic and *non-negative* in $S(0,R)$ and continuous on its closure, $\bar{S}(0,R)$. Let $\xi \in S(0,R)$ and note that

$$R - \|\xi\| \le \|x - \xi\| \le R + \|\xi\|\,, \quad x \in \partial S(0,R)\,.$$

From these estimates we obtain

$$\begin{aligned}\frac{1}{(R+\|\xi\|)^n}\int_{\partial S(0,R)} u(x)\,dS_x &\le \int_{\partial S(0,R)} \frac{u(x)}{\|x-\xi\|^n}\,dS_x \\ &\le \frac{1}{(R-\|\xi\|)^n}\int_{\partial S(0,R)} u(x)\,dS_x\,,\end{aligned}$$

and in view of (4.5) this implies

$$\left(\frac{R}{R+\|\xi\|}\right)^{n-2}\frac{R-\|\xi\|}{R+\|\xi\|}u(0) \le u(\xi) \le \left(\frac{R}{R-\|\xi\|}\right)^{n-2}\frac{R+\|\xi\|}{R-\|\xi\|}u(0)\,. \tag{1}$$

These are the *Harnack inequalities* for non-negative harmonic functions. In addition to providing estimates for the growth of non-negative harmonic functions, they imply the following.

**Proposition 3: Monotone Convergence.** *Let $u_k$ be a non-decreasing sequence of functions harmonic in the open and connected set $G$. If the sequence converges at some point of $G$, then it converges uniformly on every compact subset of $G$ to a function which is harmonic in $G$.*

*Proof.* Let $y$ be a point at which the sequence converges. Since the class of harmonic functions is invariant under a translation of coordinates, we may assume that $y = 0$. Choose $0 < r < R$ with the closure of $S(0,R)$ inside $G$. Then from the Harnack inequality (1) applied to the non-negative function $u_j - u_k\,,\ j \ge k \ge 1$, we have

$$0 \le u_j(\xi) - u_k(\xi) \le (\frac{R}{R-\|\xi\|})^{n-2}\frac{R+\|\xi\|}{R-\|\xi\|}(u_j(0) - u_k(0))\,, \quad \xi \in S(0,r)\,.$$

Since $\{u_k(0)\}$ converges, this shows that the sequence $\{u_k\}$ is uniformly Cauchy, hence, uniformly convergent in a sphere centered at 0. That is, if $u_k(y)$ converges for some $y \in G$, then $\{u_k\}$ converges uniformly in every sphere centered at $y$ whose closure is contained in $G$.

Suppose $z$ is a point in $G$ at which the sequence does *not* converge. Then the sequence cannot converge at any point in $S(z,r)$, where $r$ is half the distance from

$z$ to the boundary of $G$, since that would imply convergence at $z$ by the preceding paragraph. Thus, we have shown that the subset of $G$ on which $\{u_k\}$ diverges is open. We showed above that the subset on which $\{u_k\}$ converges is open. $G$ is connected, and the latter set is non-empty by assumption, so the former must be empty. The assertion about uniform convergence follows, since every compact set can be covered by a finite number of spheres whose closures are contained in $G$. Proposition 2 shows that the limit is harmonic.

**Exercises.** 1. Use the Mean Value Property of harmonic functions to show that the uniform limit of harmonic functions is harmonic.

2. Show that if a sequence of harmonic functions converges uniformly on every compact subset of the open set $G$, then any derivative of the sequence converges uniformly on every compact subset to the corresponding derivative of the limit function.

3. Verify that the only bounded harmonic functions on $\mathbb{R}^n$ are the constant functions.

4. Prove the Reflection Principle.

**6. The Dirichlet Problem.**

Our objective here is to show that the Dirichlet problem for Laplace's equation has a solution whenever the domain is sufficiently smooth. In fact, we shall characterize those domains for which the problem is always solvable as those for which a much simpler problem is always solvable, namely, that for each point on the boundary, there is a harmonic function which vanishes at that point and is strictly positive elsewhere in the domain. Such a function can usually be found by inspection.

The idea of the proof below is simple. Note that if $u$ is the solution to the Dirichlet problem

$$\Delta u(x) = 0,\ x \in G, \quad u(y) = g(y),\ y \in \partial G,$$

and if $v$ is any subharmonic function on $G$ with $v \le g$ on $\partial G$, then $v \le u$ on all of $\bar{G}$. Thus the solution is characterized as the 'largest' such function.

**Proposition 1.** *Let $G$ be a bounded domain in $\mathbb{R}^n$ and $u$ a subharmonic function on $G$. Let $S$ be a sphere with closure $\bar{S}$ contained in $G$. Define the function $v \in C(\bar{G})$ by $v(x) = u(x),\ x \in G - S$, and $v$ is harmonic in $S$. (Thus, $v$ is given by Poisson's formula in $S$.) Then $v$ is subharmonic in $G$ and $u \le v$ in $G$.*

*Proof.* Clearly $v$ is subharmonic in $S$ and in $G - \bar{S}$. Since $u - v$ is subharmonic in $S$ and is zero on $\partial S$, we have $u \le v$ in $S$, hence, in all of $G$. Finally, to see that $v$ is subharmonic at each $\xi \in \partial S$, we have for sufficiently small $r > 0$

$$v(\xi) = u(\xi) \le \frac{1}{\omega_n r^{n-1}} \int_{\partial S(\xi,r)} u(x)\, dS_x \le \frac{1}{\omega_n r^{n-1}} \int_{\partial S(\xi,r)} v(x)\, dS_x\,.$$

**Definition.** The function $v$, determined in Proposition 1 from the subharmonic function $u$, is called the *$S$-harmonization* of $u$, and it is denoted by $u^S$.

Let the function $g \in C(\partial G)$ be given, and define the set of functions

$$C_g \equiv \{v \in C(\bar{G}) : v \text{ is subharmonic in } G, v \le g \text{ on } \partial G\}.$$

This set is non-empty, since it contains the constant function $c = \min_{\partial G} g$. By the Maximum Principle, we have at each $x \in G$

$$\sup_{v \in C_g} v(x) \le \max_{\partial G} g$$

so we can define

$$u(x) \equiv \sup_{v \in C_g} v(x), \quad x \in G.$$

**Proposition 2.** *The function $u$ is harmonic in $G$.*

*Proof.* Let $S$ be a sphere with $\bar{S}$ in $G$, and let $x \in S$. By definition, there is a sequence $\{u_n\}$ in $C_g$ for which $\lim u_n(x) = u(x)$. Since we can replace each $u_n$ by $\max\{u_1, u_2, \ldots, u_n\}$, we may assume with no loss of generality that the sequence is monotone. (Note that the maxima of subharmonic functions is subharmonic: see Exercise 3.3.) Consider the monotone increasing sequence $\{u_n^S\}$. This sequence necessarily converges to a harmonic function $U$ in $S$, and we have $U \le u$ in $G$. We will show that $U \ge u$ in $S$. Suppose there is a $y \in S$ for which $U(y) < u(y)$. Pick a sequence $\{v_n\}$ in $C_g$ with $\lim v_n(y) = u(y)$. Set $V_n = \max\{u_1, v_1, u_2, v_2, \ldots, u_n, v_n\}$, so that the sequence $\{V_n\}$ is increasing in $G$, $\lim V_n(x) = u(x)$ and $\lim V_n(y) = u(y)$. Consider the sequence of $S$-harmonizations: $\{V_n^S\}$. As before, this sequence converges upward to a function $V$ harmonic in $S$, and we have

$$V \ge U,\ V(x) = U(x) = u(x),\ V(y) = u(y) > U(y).$$

By the Maximum Principle, the first two imply that $V = U$ in $S$, and this contradicts the third. Thus, we have shown that $u = U$ in $S$, so $u$ is harmonic (in each sphere contained) in $G$.

Now we consider the boundary values of the limiting function, $u$. For this, it will be useful to introduce the following.

**Definition.** Let $G$ be a bounded domain in $\mathbb{R}^n$. For any point $y \in \partial G$, a *barrier* for $G$ at $y$ is a function $w \in C(\bar{G})$, harmonic in $G$, with $w(y) = 0$ and $w(x) > 0$ for all $x \in \bar{G} - \{y\}$. The boundary point $y$ is called *regular* if there exists a barrier for $G$ at $y$.

For example, if there is a sphere $S(\xi, r)$ for which $\bar{G} \cap S(\xi, r)$ is empty but $\bar{G} \cap \bar{S}(\xi, r) = \{y\}$, then we can use the singularity function to construct a barrier for $G$ at $y$ by

$$w(x) = s(y, \xi) - s(x, \xi),$$

so such a point $y$ is regular. In particular, if the boundary $\partial G$ is $C^2$, then every boundary point is regular.

**Proposition 3.** *If the point $y \in \partial G$ is regular, then the limiting function satisfies*

$$\lim_{\substack{x \in G \\ x \to y}} u(x) = g(y).$$

*Proof.* Let $w$ be a barrier for $G$ at $y$. Fix $\varepsilon > 0$ and choose $r > 0$ so small that $S(y, r)$ does not cover all of $G$ and for all $x \in \partial G$ with $\|x - y\| < r$, we have

$|g(x) - g(y)| < \varepsilon$. Note that necessarily $w_0 \equiv \min_{x \notin S(y,r)}\{w(x)\} > 0$. Consider the harmonic function

$$v_0(x) = \varepsilon + g(y) + \frac{\max_{\partial G} g - g(y)}{w_0} w(x), \quad x \in \bar{G}.$$

It follows that $v_0 \geq g(y) + \varepsilon$ in $\bar{G}$ and so for $x \in \partial G \cap S(y,r)$ we have $v_0(x) > g(x)$. Also, we have for those $x \in \partial G - S(y,r)$

$$v_0(x) \geq \varepsilon + g(y) + (\max_{\partial G} g - g(y)) > g(x)\,,$$

so $g < v_0$ everywhere on $\partial G$. Now for any $v \in C_g$ if follows from the Maximum Principle that $v < v_0$ in $G$, so the limiting function satisfies

$$u(x) \leq v_0(x), \quad x \in G.$$

This gives the estimate

$$\limsup_{x \to y} u(x) \leq \limsup_{x \to y} v_0(x) = \lim_{x \to y} v_0(x) = \varepsilon + g(y)\,,$$

hence, we have $\limsup_{x \to y} u(x) \leq g(y)$. Similarly, consider the harmonic function

$$v_0(x) = -\varepsilon + g(y) + \frac{\min_{\partial G} g - g(y)}{w_0} w(x), \quad x \in \bar{G}.$$

As above, we find that $v_0 \in C_g$ and, hence, $v_0 \leq u$ in $G$. This gives

$$\liminf_{x \to y} u(x) \geq \liminf_{x \to y} v_0(x) = \lim_{x \to y} v_0(x) = -\varepsilon + g(y)\,,$$

so we have $\liminf_{x \to y} u(x) \geq g(y)$. That is, we have shown that $\lim_{x \to y} u(x) = g(y)$.

**Theorem: Perron's Method.** *Let $G$ be a bounded domain for which each boundary point is regular. Then for each boundary function $g \in C(\partial G)$ there exists a unique function $u \in C(\bar{G})$ which is harmonic in $G$ and for which $u = g$ on $\partial G$.*

We note finally, that if the Dirichlet problem can be solved for any countinuous boundary function, then each point $y \in \partial G$ is necessarily regular. It is only necessary to solve the Dirichlet problem with $g(x) = \|x - y\|$ for $x \in \partial G$.

**Theorem.** *If $G$ is a bounded domain and $f \in C^1(\bar{G})$, then*

$$u(x) = -\int_G s(x,y) f(y)\, dy$$

*belongs to $C(\bar{G}) \cap C^2(G)$ and satisfies $-\Delta u(x) = f(x)$ in $G$.*