

ZEROS OF CLASSICAL EISENSTEIN SERIES AND RECENT DEVELOPMENTS

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ABSTRACT. In this survey, we begin by recalling a beautiful result of F. K. C. Rankin and Swinnerton-Dyer on the location of zeros of the classical Eisenstein series $E_k(z)$ for the full modular group. We then explore more recent studies which have built upon this work to analyze the behavior of zeros of $E_k(z)$, such as work by Nozaki on their separation property. We also review similar results for other classes of modular forms as well as zeros of Eisenstein series for different groups. We conclude with some results of the authors for a family of odd weight Eisenstein series on $\Gamma(2)$, a prototypical genus zero subgroup with a simple fundamental domain.

1. EISENSTEIN SERIES ON $\mathrm{SL}_2(\mathbb{Z})$

We begin this survey by defining modular forms on the full modular group $\Gamma := \mathrm{SL}_2(\mathbb{Z})$. Let \mathbb{H} denote the complex upper half-plane

$$\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}.$$

The modular group Γ acts on \mathbb{H} , or more generally $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, by linear fractional transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z := \frac{az + b}{cz + d}.$$

As the two matrices $\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ generate the same action, it is also sometimes appropriate to restrict our view to $\mathrm{PSL}_2(\mathbb{Z}) := \mathrm{SL}_2(\mathbb{Z})/\{\pm I\}$. A fundamental domain for this action is given by

$$\mathcal{F} := \{z \in \mathbb{H} : -1/2 \leq \mathrm{Re}(z) \leq 1/2, |z| \geq 1\}.$$

We define a modular form f of integer weight k for Γ to be a holomorphic function on \mathbb{H} such that

$$f\left(\frac{az + b}{cz + d}\right) = (cz + d)^k f(z) \tag{1.1}$$

for all $z \in \mathbb{H}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, and such that f is holomorphic when extended to ∞ .

Note that under the action of Γ the points $\mathbb{Q} \cup \{\infty\}$ form one equivalence class; we often refer to these points collectively as the cusp of Γ . Alternatively, we call an element $\gamma \in \Gamma$ parabolic if γ has only one fixed point on

$\widehat{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$, and define the cusps to be the set of equivalence classes of fixed points of the parabolic elements. In Γ , and in fact in $\mathrm{SL}_2(\mathbb{R})$, the parabolic elements are exactly those with trace ± 2 .

The above conditions imply that a modular form f has a Fourier expansion of the form

$$\sum_{n \geq 0} a(n)q^n, \quad (1.2)$$

where here and throughout this paper we define $q := e^{2\pi iz}$. Moreover, the set of modular forms of fixed weight k is a finite-dimensional vector space over \mathbb{C} , denoted M_k . If $f \in M_{k_1}$ and $g \in M_{k_2}$ then $fg \in M_{k_1+k_2}$. Hence, the collection of all positive integral weight modular forms is a graded algebra. For more basic properties of modular forms, see ([Ser73], [Kob93], [Ono04]).

Of primary importance among modular forms for the full modular group are the Eisenstein series. For even weight $k \geq 4$, the functions

$$G_k(z) := \sum_{\substack{c,d \in \mathbb{Z} \\ (c,d) \neq (0,0)}} (cz + d)^{-k} \quad (1.3)$$

are holomorphic modular forms of weight k . These classical Eisenstein series can be viewed as arising from the Weierstrass \wp function, an elliptic function satisfying the differential equation

$$(\wp'(u))^2 = 4\wp(u)^3 - 60G_4\wp(u) - 140G_6. \quad (1.4)$$

One can solve this differential equation recursively to recover \wp as a generating function for the G_k :

$$\wp(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty} (2k+1)G_{2k+2}z^{2k}.$$

The series $G_k(z)$ are often normalized to have a constant coefficient of 1 by dividing by $2\zeta(k)$. Alternatively, one can write these normalized Eisenstein series, denoted $E_k(z)$, as

$$E_k(z) := \frac{1}{2} \sum_{\substack{c,d \in \mathbb{Z} \\ (c,d)=1}} (cz + d)^{-k}. \quad (1.5)$$

There are many questions that naturally arise about Eisenstein series and modular forms in general. For example, one might be interested in the number of zeros that these functions have and their location, which is the focus of this survey. A key tool in the study of zeros of modular forms is the valence formula. Define $\nu_\tau(f)$ to be the order of a modular form f at a point $\tau \in \widehat{\mathbb{H}}$, where $\widehat{\mathbb{H}} := \mathbb{H} \cup \{\infty\} \cup \mathbb{Q}$. The valence formula states that if f is a nonzero modular form of weight k , then

$$\nu_\infty(f) + \frac{1}{2}\nu_i(f) + \frac{1}{3}\nu_\omega(f) + \sum_{\substack{\tau \in \Gamma \backslash \mathbb{H} \\ \tau \neq i, \omega}} \nu_\tau(f) = \frac{k}{12}, \quad (1.6)$$

where i and ω are the usual primitive fourth and third roots of unity in \mathbb{H} .

We call an element $\gamma \in \Gamma$ elliptic if $\gamma \cdot z = z$ has exactly one solution in \mathbb{H} ; we refer to these fixed points as elliptic points. The elliptic elements are those with trace strictly less than 2 in absolute value, and, along with $\pm I$, are the elements of Γ with finite order. The corresponding elliptic points are Γ -equivalent to either i or ω , which are of order 2 and 3, respectively.

Notice that equation (1.6) implies that the number of zeros of a modular form is roughly one-twelfth of its weight. In addition, it follows from (1.6) that for $k \geq 4$, the set

$$\{E_4^a E_6^b \mid 4a + 6b = k \text{ and } a, b \geq 0\}$$

forms a basis for M_k . This fact illustrates the fundamental role Eisenstein series play as building blocks in the theory of modular forms.

2. A CLASSICAL RESULT OF RANKIN AND SWINNERTON-DYER

The study of the location of zeros of Eisenstein series has its origins in work of Wohlfahrt and R. A. Rankin in the 1960s, who showed in [Woh63] and [Ran68] that for even $4 \leq k \leq 38$ (excepting $k = 36$), the zeros of $E_k(z)$ lie on the unit circle. R. A. Rankin also gave partial results towards establishing this property for all $k \geq 4$. Soon after, R. A. Rankin's daughter F. K. C. Rankin, together with Swinnerton-Dyer [RSD70], used an elementary yet elegant argument to prove the following.

Theorem 2.1. *For even $k \geq 4$, all zeros of $E_k(z)$ in the fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$ are located on the arc*

$$A := \{z = e^{i\theta} : \pi/2 \leq \theta \leq 2\pi/3\}.$$

Here we give a brief outline of their argument. Write $k = 12n + s$ with $s = 4, 6, 8, 10, 0$, or 14 . Note, by the valence formula (1.6), that the value s determines the minimum number of zeros that $E_k(z)$ must have at the elliptic points i and ω . Thus, it is sufficient to show that $E_k(e^{i\theta})$ has at least n zeros in the open interval $(\pi/2, 2\pi/3)$. For $n > 0$, F. K. C. Rankin and Swinnerton-Dyer consider

$$F_k(\theta) = e^{ik\theta/2} E_k(e^{i\theta}) = 2 \cos(k\theta/2) + R, \quad (2.1)$$

where R is the remainder term consisting of all terms in (1.5) with $c^2 + d^2 > 1$. They show that $|R| < 2$ on $[\pi/2, 2\pi/3]$, and hence the number of zeros of $E_k(e^{i\theta})$ in $(\pi/2, 2\pi/3)$ is at least the number of zeros of $2 \cos(k\theta/2)$ on the same interval; a short calculation reveals that there are n of them.

Theorem 2.1 can also be framed in terms of the j -function,

$$j(z) := 1728 E_4^3(z) / (E_4^3(z) - E_6^2(z)), \quad (2.2)$$

which is an isomorphic map between \mathcal{F} and \mathbb{C} . For example, $j(i) = 1728$ and $j(\omega) = 0$. This function is invariant under Γ , with equivalent boundary points identified, and holomorphic on \mathbb{H} , with a simple pole at the cusp ∞ . In this regard it is a weakly holomorphic modular form; weakly holomorphic

modular forms satisfy (1.1) and (1.2), except that finitely many terms with $n < 0$ are allowed in the latter. They form an infinite-dimensional vector space, denoted \mathcal{M}_k .

The j -function generates the function field of the genus zero modular curve $X(\Gamma) := \Gamma \backslash \widehat{\mathbb{H}}$. It also parameterizes isomorphism classes of elliptic curves. Theorem 2.1 is equivalent to the statement that the j -values of the zeros of $E_k(z)$ are all real and lie within the interval $[0, 1728]$.

3. RECENT RESULTS FOR $\mathrm{SL}_2(\mathbb{Z})$

Following the work of F. K. C. Rankin and Swinnerton-Dyer there have been more recent discoveries about the zeros of the classical Eisenstein series, as well as several generalizations to other classes of modular forms. In this section we investigate some recent results about zeros of modular forms on $\mathrm{SL}_2(\mathbb{Z})$.

In [Koh04] Kohlen derives a closed formula for the precise locations of the zeros of $E_k(z)$ on A , as defined in Theorem 2.1. The exact statement of his result requires additional notation, which we omit for brevity. We do note, however, that the formula is an infinite sum whose terms are related to the Fourier coefficients of $E_k(z)$; in particular, they include Bernoulli numbers and divisor sums.

In [Gek01], Gekeler proves p -adic congruence properties for the following polynomials which encode the j -values of the non-elliptic zeros of $E_k(z)$ for even $k \geq 4$:

$$\varphi_k(x) = \prod_{\substack{j(z) \text{ where } E_k(z)=0 \\ j(z) \neq 0, 1728}} (x - j(z)). \quad (3.1)$$

Part of the interest in studying these polynomials is that for primes $p \geq 5$, φ_{p-1} is a canonical lift of essentially the supersingular polynomial in characteristic p , whose zeros are the j -invariants of the supersingular elliptic curves defined over $\overline{\mathbb{F}}_p$ (see e.g. [Ono04]). Gekeler shows that up to a constant multiple, the reduction of φ_k modulo p contains φ_{p-1} as a factor.

Gekeler also cites empirical evidence computed by Kremling [Krem99] that suggests that the $\varphi_k(x)$ are irreducible with full Galois group S_d where d is the degree of $\varphi_k(x)$. Moreover, the zeros of φ_{k+12a} for $a \in \mathbb{Z}_{\geq 0}$ indicate separation properties one would expect from orthogonal polynomials, although they are not orthogonal.

In addition, by studying the coefficients of $\varphi_k(x)$ it is possible to relate the location of the zeros of $E_k(z)$ to special values of the Riemann zeta function $\zeta(s)$. The resulting formula

$$\frac{2}{\zeta(1-k)} = 60k - \sum_{\tau \in \Gamma \backslash \mathbb{H}} e_\tau \mathrm{ord}_\tau(E_k(\tau)) j(\tau), \quad (3.2)$$

where $e_i = 1/2$, $e_\omega = 1/3$, and $e_\tau = 1$ for non-elliptic points τ , is highlighted in [OP04].

Recently, Nozaki [Noz08] has proved the separation property observed by Gekeler in [Gek01]. His precise statement is as follows.

Theorem 3.1. *Let $k \geq 12$ be an even integer, and let $E_k(z)$ be the Eisenstein series with weight k for $\mathrm{SL}_2(\mathbb{Z})$. Let $\{e^{ia_j} \mid a_1 < a_2 < \dots < a_n\}$ be the zeros of $E_k(z)$ on A and let $\{e^{ib_j} \mid b_1 < b_2 < \dots < b_{n+1}\}$ be the zeros of $E_{k+12}(z)$ on A . Then $b_j < a_j < b_{j+1}$ for $j = 1, 2, \dots, n$.*

To prove this result, Nozaki considers $F_k(z)$ as defined in (2.1). In [RSD70], F. K. C. Rankin and Swinnerton-Dyer proved that it was possible to approximate the location of the zeros of F_k and F_{k+12} by the location of the zeros of $2 \cos(k\theta/2)$ and $2 \cos((k+12)\theta/2)$, respectively. By considering the decreasing size of the main error terms in the argument more closely, Nozaki is able to prove that in fact the zeros of F_k and F_{k+12} do not stray far from the zeros of their associated trigonometric functions. In particular, they stray by less than half the distance of the two closest zeros of $2 \cos(k\theta/2)$ and $2 \cos((k+12)\theta/2)$.

Shifting focus to another family, Asai, Kaneko and Ninomiya consider the zeros of a family of weakly holomorphic modular forms which are related to the j -function via the Hecke operators. Let T_n be the n th Hecke operator, which acts on a modular form $f(z)$ of weight k by

$$f(z) |_k T_n = n^{k-1} \sum_{ad=n} \sum_{b=0}^{d-1} d^{-k} f\left(\frac{az+b}{d}\right)$$

(see, e.g. [Ser73]). Define

$$J_n(z) := n(j(z) - 744) |_0 T_n.$$

It is well known that $J_n(z)$ is a monic polynomial in $j(z)$ of degree n . Asai, et al. prove the following in [AKN97], where A is defined as in Theorem 2.1.

Theorem 3.2. *For each $n \geq 1$, all the zeros of $J_n(z)$ are simple and lie on the arc A .*

Equivalently, the zeros of the polynomial $J_n(j)$ lie in the interval $(0, 1728)$. Their method is similar to the F. K. C. Rankin and Swinnerton-Dyer method in that they show that the values of $J_n(z)$ do not stray far from the values of a cosine function, and hence the number of zeros may be easily counted.

Gun similarly extends the results of F. K. C. Rankin and Swinnerton-Dyer to another class of modular forms. In [Gun06], Gun examines zeros for a particular class of cusp forms, which are holomorphic modular forms that vanish at ∞ . To state his result, we first define the stabilizer of ∞ ,

$$\Gamma_\infty := \{\gamma \in \Gamma : \gamma \cdot \infty = \infty\} = \left\{ \pm \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} : m \in \mathbb{Z} \right\}.$$

In addition, we make the following definition.

Definition 3.3. For a fixed even $k \geq 4$, write $k = 12n + s$, where $s \in \{0, 4, 6, 8, 10, 14\}$, as in the proof of Theorem 2.1. For $1 \leq j \leq n-1$, let

$f_{k,j}(z)$ denote any sum

$$f_{k,j}(z) = \sum_{\ell=1}^j a(\ell)P_{\ell}(z)$$

where P_{ℓ} is the ℓ th Poincare series

$$P_{\ell}(z) = \sum_{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_{\infty} \setminus \Gamma} (cz + d)^{-k} e^{2\pi i \ell(\gamma \cdot z)},$$

and the $a(\ell)$ are real numbers with $a(\ell) \geq 0$ for $1 \leq \ell \leq j-1$ and $a(j) > 0$. The $f_{k,j}$ are cusp forms of weight k .

Gun extends results of R. A. Rankin [Ran82] on the Poincare series $P_{\ell}(z)$ to prove the following.

Theorem 3.4. *For even $k \geq 26$, $f_{k,j}(z)$ has at least $n - j$ zeros on the interior of A .*

Moreover, following Kohnen [Koh04], he obtains a closed formula for the zeros of $f_{k,1}$ showing that all zeros lie on A except for a simple zero at infinity.

In [DJ08], Duke and Jenkins provide a result on the location of zeros of a class of weakly holomorphic modular forms that unifies some of the above results. The modular forms they investigate are a natural basis of the vector space \mathcal{M}_k of weakly holomorphic modular forms of weight k . In addition, the coefficients of these forms exhibit an intriguing duality property. These basis elements are defined as follows. Let n be defined as in the proof of Theorem 2.1 and Definition 3.3. For each integer $m \geq -n$, there exists a unique $f_{k,m} \in \mathcal{M}_k$ with q -expansion given by

$$f_{k,m}(z) = q^{-m} + O(q^{n+1}).$$

Any weakly holomorphic modular form $f(z) = \sum a(m)q^m \in \mathcal{M}_k$ can be written as

$$f(z) = \sum_{m_0 \leq m \leq n} a(m)f_{k,-m}.$$

This basis $\{f_{k,m} \mid m \geq -n\}$ can be written explicitly in terms of the Eisenstein series $E_s(z)$, $j(z)$, and $\Delta(z) := (E_4^3(z) - E_6^2(z))/1728$, and thus can be shown to have integer coefficients. The following theorem is given in [DJ08].

Theorem 3.5. *Using the notation above, if $m \geq |n| - n$, then all of the zeros of $f_{k,m}$ in \mathcal{F} lie on the unit circle.*

One should note that the restriction that $m \geq 0$ excludes cusp forms from the scope of Theorem 3.5. In particular Duke and Jenkins provide examples to show that the result does not hold for general m . However, the results of F. K. C. Rankin and Swinnerton-Dyer in [RSD70], R. A. Rankin in [Ran82], and Asai, Kaneko, and Ninomiya in [AKN97] all deal with certain subclasses of the $\{f_{k,m}\}$. Duke and Jenkins depart somewhat from the method of proof

first employed by F. K. C. Rankin and Swinnerton-Dyer in that they use a circle method argument to bound the error term expressing the distance between their function and an appropriate cosine function. The special case $m = 0$ was previously studied by Getz using an alternative method. See [Get04, Get10].

4. EISENSTEIN SERIES ON OTHER GROUPS

We can expand our definition of modular forms to include functions which transform as in (1.1) for groups other than Γ . A natural generalization is to consider certain subgroups of Γ . In this section we will consider some examples of such variations.

Let $N \geq 1$ be an integer, and define

$$\Gamma(N) := \{M \in \Gamma : M \equiv I \pmod{N}\},$$

where the congruence is taken componentwise. A congruence subgroup of level N is any group Γ' such that $\Gamma(N) < \Gamma' < \Gamma$. A modular form f of weight k for Γ' is a holomorphic function on \mathbb{H} which satisfies (1.1) for all $z \in \mathbb{H}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma'$. In addition, f is holomorphic when extended to the cusps of Γ' . There are finitely many equivalence classes of cusps modulo Γ' . These conditions imply that f has a Fourier expansion at each cusp, of the form

$$\sum_{n \geq 0} \alpha(n) q_N^n, \quad q_N := e^{2\pi iz/N}.$$

We denote the finite-dimensional complex vector space of these modular forms by $M_k(\Gamma')$.

One can define Eisenstein series associated to each cusp of a congruence subgroup Γ' . For simplicity, we only define the Eisenstein series associated with ∞ ; for further details about Eisenstein series of level N see [DS05].

Let $\Gamma'_\infty := \{\gamma \in \Gamma' : \gamma \cdot \infty = \infty\}$. Then

$$E_k(\Gamma'; z) := \sum_{\gamma \in \Gamma'_\infty \backslash \Gamma'} (cz + d)^{-k},$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Note that this matches (1.5) for $\Gamma' = \Gamma$.

To extend the results of the previous section on the location of zeros of Eisenstein series, it is natural to look for other groups which share some nice properties of Γ . In particular, the Riemann surface associated to Γ has genus zero, and its fundamental domain is quite simple. The principal congruence subgroup $\Gamma(2)$ also has genus zero and a simple fundamental domain. In [GLST], the authors extend Theorems 2.1 and 3.1 to certain odd weight Eisenstein series for $\Gamma(2)$. We review these results in section 5.

One can also consider modular forms on Fricke groups. For p prime, define

$$\Gamma_0^*(p) := \Gamma_0(p) \cup \Gamma_0(p)W_p, \quad W_p := \begin{pmatrix} 0 & -1/\sqrt{p} \\ \sqrt{p} & 0 \end{pmatrix}.$$

One can define modular forms for $\Gamma_0^*(p)$ analogously to modular forms for congruence subgroups. Note that $\Gamma_0^*(p)$ is not a subgroup of Γ , but is commensurable with Γ , meaning that the intersection $\Gamma \cap \Gamma_0^*(p)$ has finite index in both groups.

Define

$$E_{k,p}^*(z) := \frac{1}{p^{k/2} + 1} (p^{k/2} E_k(pz) + E_k(z))$$

to be the Eisenstein series associated to $\Gamma_0^*(p)$, where $E_k(z)$ is the normalized Eisenstein series for Γ . The location of zeros of $E_{k,p}^*(z)$ has been investigated for $p = 2$ and 3 by Miezaki, Nozaki and Shigezumi in [MNS07], and for $p = 5$ and 7 by Shigezumi in [Shi07]. In each case, the F. K. C. Rankin and Swinnerton-Dyer method extends by a careful analysis of an error term $F_{k,p}^*(\theta)$, which is an appropriate analog of $F_k(\theta)$ as defined in (2.1).

For $p = 2$ and 3 , a fundamental domain for $\Gamma_0^*(p)$ is given by

$$\mathcal{F}^*(p) := \{z \in \mathbb{H} : -1/2 \leq \operatorname{Re}(z) \leq 1/2, |z| \geq 1/\sqrt{p}\}.$$

Miezaki, Nozaki and Shigezumi prove the following in [MNS07].

Theorem 4.1. *Let $p = 2$ or 3 , and set*

$$A_p^* := \left\{ z = e^{i\theta}/\sqrt{p} : \pi/2 \leq \theta \leq \frac{\pi(2p-1)}{2p} \right\}.$$

Then for even $k \geq 4$, all the zeros of $E_{k,p}^(z)$ in $\mathcal{F}^*(p)$ are on the arc A_p^* .*

For the cases $p = 5$ and 7 , the fundamental domain is a bit more complicated and defined as follows. Let

$$\begin{aligned} \mathcal{F}^*(p) := & \{z \in \mathbb{H} : -1/2 \leq \operatorname{Re}(z) \leq 0, |z| \geq 1/\sqrt{p}, |z + 1/2| \geq 1/(2\sqrt{p})\} \\ & \cup \{z \in \mathbb{H} : 0 \leq \operatorname{Re}(z) \leq 1/2, |z| \geq 1/\sqrt{p}, |z - 1/2| \geq 1/(2\sqrt{p})\}. \end{aligned}$$

Moreover, let α_5 and α_7 be the angles which satisfy $\tan \alpha_5 = 2$ and $\tan \alpha_7 = 5/\sqrt{3}$, respectively, and define

$$A_5^* := \{e^{i\theta}/\sqrt{5} : \pi/2 \leq \theta \leq \pi/2 + \alpha_5\} \cup \{-1/2 + e^{i\theta}/2\sqrt{5} : \alpha_5 < \theta \leq \pi/2\},$$

and

$$A_7^* := \{e^{i\theta}/\sqrt{7} : \pi/2 \leq \theta \leq \pi/2 + \alpha_7\} \cup \{-1/2 + e^{i\theta}/2\sqrt{7} : \alpha_7 - \pi/6 < \theta \leq \pi/2\}.$$

Shigezumi conjectures the following in [Shi07].

Conjecture. *For $p = 5$ or 7 and for even $k \geq 4$, all the zeros of $E_{k,p}^*(z)$ in $\mathcal{F}^*(p)$ lie on the arc A_p^* .*

The difficulty in proving this conjecture arises from considering the two distinct arcs that form \mathcal{A}_p^* ; Shigezumi is able to modify the method of F. K. C. Rankin and Swinnerton-Dyer on each section, but has difficulty bounding the error term of $F_{k,p}^*(\theta)$ at $\theta = \pi/2 + \alpha_p$, the point of intersection of the two arcs. Moreover, whereas F. K. C. Rankin and Swinnerton-Dyer find it sufficient to bound their error term in absolute value, Shigezumi

demonstrates that the sum of the absolute values of the first few terms in his errors already exceed the necessary upper bound. He instead considers the arguments of these terms to improve the bound. His approach fails when $k(\pi/2 + \alpha_p)/2 \pmod{\pi}$ lies in a certain narrow range of values. However, Shigezumi is able to achieve the following results, which we summarize as follows.

Proposition 4.2. *Let $p = 5$ or 7 . All but at most one of the zeros of $E_{k,p}^*(z)$ in $\mathcal{F}^*(p)$ lie on A_p^* . In addition, if $p = 5$ and $4 \mid k$, or $p = 7$ and $6 \mid k$, then all of the zeros of $E_{k,p}^*(z)$ in $\mathcal{F}^*(p)$ lie on A_p^* .*

We now turn to a general result about the location of the zeros of Eisenstein series. So far we have mainly been interested in subgroups of $\mathrm{SL}_2(\mathbb{Z})$. We can naturally extend our definitions of linear fractional transformations, cusps, and modular forms to subgroups of $\mathrm{SL}_2(\mathbb{R})$, or similarly $\mathrm{PSL}_2(\mathbb{R})$, with the Fricke subgroups as an example.

We define a norm on $\mathrm{SL}_2(\mathbb{R})$ by

$$\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \| := a^2 + b^2 + c^2 + d^2,$$

which gives the group a metric topology. We consider a subgroup Γ' of $\mathrm{SL}_2(\mathbb{R})$ to be discrete if for any $\epsilon > 0$ the set $\{\gamma \in \Gamma' : \|\gamma\| < \epsilon\}$ is finite. Alternatively, a subgroup of $\mathrm{SL}_2(\mathbb{R})$ is discrete if and only if it acts discontinuously on \mathbb{H} when viewed as a subgroup of $\mathrm{PSL}_2(\mathbb{R})$. We call such a subgroup a Fuchsian group. Note that as it acts discontinuously on \mathbb{H} , the orbits $\Gamma' \cdot z = \{\gamma \cdot z : \gamma \in \Gamma'\}$ for each $z \in \mathbb{H}$ have no limit points in \mathbb{H} ; however, there may be limit points on the boundary $\widehat{\mathbb{R}}$. More specifically, we call a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ a Fuchsian group of the first kind if every point on $\widehat{\mathbb{R}}$ is in the set of limit points of the orbit $\Gamma' \cdot z$ for some $z \in \mathbb{H}$. For a more complete discussion of modular forms on Fuchsian groups, see [Iwa97] or [Sh71].

Hahn [Hah07] provides a general result about the zeros of Eisenstein series for certain genus zero Fuchsian groups of the first kind. She considers, for such a group Γ' with a cusp at ∞ , a specific Eisenstein series $E_k(\Gamma') \in M_k(\Gamma')$ (the vector space of holomorphic modular forms on Γ') meeting certain technical conditions. Among these conditions is that $E_k(\Gamma')$ must vanish at all cusps but ∞ , have real Fourier coefficients, and have 1 as its constant term. In addition, the corresponding fundamental domains must meet acceptable conditions, including that the Hauptmodul $j_{\Gamma'}$ associated to Γ' is real on the boundary. For details, see [Hah07].

In general, Hahn's results are slightly weaker than the aforementioned results; however her argument, involving divisor polynomials similar to (3.1), applies to a large family of groups.

5. A FAMILY OF EISENSTEIN SERIES ON $\Gamma(2)$

In this section we explore in greater depth the location and interlacing property of zeros of a certain family of odd weight Eisenstein series on the

congruence subgroup $\Gamma(2)$. While we expect that Eisenstein series on other congruence subgroups behave similarly, we choose to focus on $\Gamma(2)$ because in this setting the Eisenstein series have an elegant connection to an elliptic function $cn(u)$ which plays a role analogous to that of $\wp(n)$ for Γ . The following results arose from the collaboration by the authors initiated at the WIN conference.

We consider the congruence subgroup $\Gamma(2)$, where

$$\Gamma(2) := \{M \in \mathrm{SL}_2(\mathbb{Z}) : M \equiv I \pmod{2}\}.$$

For $0 \leq x, y \leq 3$, we define

$$S_k(x, y) = S_k(x, y)(z) := \sum_{\substack{c, d \in \mathbb{Z}, (c, d) = 1 \\ c \equiv x \pmod{4}, d \equiv y \pmod{4}}} \frac{1}{(cz + d)^k}.$$

Then for integers $k \geq 0$ let

$$E_{2k+1, \chi} := \frac{1}{2}(S_{2k+1}(0, 1) + S_{2k+1}(2, 1) - S_{2k+1}(0, 3) - S_{2k+1}(2, 3)). \quad (5.1)$$

Each $E_{2k+1, \chi}(z)$ is a weight $2k + 1$ modular form on $\Gamma(2)$ with character χ , where

$$\chi \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \chi_{-1}(d) := \left(\frac{-1}{d} \right),$$

and (\cdot) stands for the Jacobi symbol. This means that $E_{2k+1, \chi}(z)$ satisfies

$$E_{2k+1, \chi} \left(\frac{az + b}{cz + d} \right) = \chi(d)(cz + d)^k E_{2k+1, \chi}(z)$$

for each $z \in \mathbb{H}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(2)$; note that this is a twisted version of (1.1).

Moreover, as in the classical case, $E_{2k+1, \chi}(z)$ has a Fourier expansion expressible in terms of generalized Bernoulli numbers and generalized divisor sums. For a Dirichlet character ψ of modulus N , the generalized Bernoulli numbers $B_{k, \psi}$ are defined by

$$\sum_{j=1}^N \frac{\psi(j) x e^{jx}}{e^{Nx} - 1} = \sum_{k=0}^{\infty} B_{k, \psi} \cdot \frac{x^k}{k!}.$$

Similarly, the generalized divisor sums $\sigma_{\psi, k}(n)$ are defined by

$$\sigma_{\psi, k}(n) := \sum_{d|n} \psi(d) d^k.$$

With this notation, the Fourier expansion of $E_{2k+1, \chi}(z)$ at ∞ is

$$1 - \frac{2(2k+1)}{B_{2k+1, \chi}} \sum_{n=1}^{\infty} \sigma_{\chi, 2k}(n) q^n.$$

We focus on this particular family for two reasons. First, $\Gamma(2)$ has some nice properties; the corresponding Riemann surface has genus zero, and $\Gamma(2)$ has a simple fundamental domain given by

$$D = \{z \in \mathbb{H} : -1 \leq \operatorname{Re}(z) \leq 1, |z - 1/2| \geq 1/2, |z + 1/2| \geq 1/2\}.$$

Note that this subgroup has three inequivalent cusps at $0, 1$, and ∞ . This is a prototypical group, and so we expect that our results generalize to similar simple genus zero congruence subgroups.

The second reason we focus on this family is that it is possible to study the location of the zeros of these series using the classical Jacobi elliptic function $cn(u)$ [Han58], which satisfies

$$\left(\frac{d cn(u)}{d u}\right)^2 = (1 - cn^2(u))(1 - \lambda + \lambda cn^2(u)). \quad (5.2)$$

Here λ is a Hauptmodul for $\Gamma(2)$; it plays a similar role to that of $j(z)$ defined in (2.2), parameterizing all isomorphism classes of elliptic curves with full 2-torsion structure, and can be defined in terms of Jacobi theta functions. Note the similarity of (5.2) to the differential equation (1.4) involving $\wp(n)$.

From results of the second author and Yang in [LY05] we obtain a recursive formula for the λ -values of the zeros of the expansion of $E_{2k+1, \chi}$ at the cusp 1 given by

$$\begin{aligned} G_{2k+1}(z) &:= E_{2k+1, \chi} |_{\gamma_0}(z) \\ &= \frac{1}{2} (S_{2k+1}(1, 0) + S_{2k+1}(3, 2) - S_{2k+1}(3, 0) - S_{2k+1}(1, 2)), \end{aligned}$$

where $\gamma_0 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$. The recursion is given in Proposition 2.3 of [GLST].

Data from the first few dozen k values indicates that all λ -values of the zeros of $G_{2k+1}(z)$ are real and are within the interval $(-\infty, 0]$, or equivalently, the zeros of $G_{2k+1}(z)$ lie on $\operatorname{Re}(z) = 1$. Moreover, this data indicates that the zeros of $G_{2k-1}(z)$ interlace with the zeros of $G_{2k+1}(z)$. Thus, there is numerical evidence that these Eisenstein series do in general satisfy the same properties proved by Rankin and Swinnerton-Dyer [RSD70] and Nozaki [Noz08] for the classical case.

In order to procure a general result, we can again permute the boundaries of the fundamental domain D , sending $\operatorname{Re}(z) = 1$ to the circular arc

$$A_2 := \left\{ z = \frac{1}{2}e^{i\theta} - \frac{1}{2} : 0 \leq \theta \leq \pi \right\},$$

by considering

$$G_{2k+1} |_{\gamma_1}(z) = \frac{1}{2} (S_{2k+1}(0, 3) + S_{2k+1}(2, 1) - S_{2k+1}(0, 1) - S_{2k+1}(2, 3)),$$

where $\gamma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. This allows us to apply the argument of Rankin and Swinnerton-Dyer to verify the location of zeros. A direct application proves that at least one third of the zeros of $G_{2k+1} |_{\gamma_1}(z)$ do indeed lie on this arc; however, we can modify the proof to do better. The main obstacle in

this case is in splitting each Eisenstein series into a main term and an error term sufficiently small, since these error terms are difficult to bound near the cusps, i.e., near $\theta = 0, \pi$.

With a more careful consideration of the error terms, it is possible to obtain the following result.

Theorem 5.1 ([GLST] Theorem 1.1). *For $k \geq 1$, at least 90% of the zeros of $G_{2k+1}(z)$ in D lie on A_2 .*

Theorem 5.1 is equivalent to the statement that at least 90% of the zeros have real λ -values in the range $(-\infty, 0]$. In addition, by restricting our domain slightly we obtain the following result about the separation property of the zeros of $G_{2k-1}(z)$ and $G_{2k+1}(z)$ following an approach of Nozaki [Noz08]. We state this result in terms of the related function of θ

$$F_{2k+1}(z_\theta) := (e^{-\frac{i\theta}{2}})^{-2k-1} G_{2k+1} |_{\gamma_1}(z_\theta),$$

where $z_\theta := \frac{1}{2}e^{i\theta} - \frac{1}{2}$, and in terms of the following intervals for integers $k > 15$ defined by

$$I_{j,2k-1} = \left(\alpha_{j,k} - \frac{2\pi}{(2k+1)(2k-1)}, \alpha_{j,k} + \frac{2\pi}{(2k+1)(2k-1)} \right)$$

for each $j = 1, \dots, k-1$, where $\alpha_{j,k} := 2\pi j / (2k-1)$. Let

$$I_{2k-1} := \bigcup_j I_{j,2k-1}.$$

We have the following result.

Theorem 5.2 ([GLST] Theorem 1.2). *Let $k > 15$ be an integer, and $I_{j,2k-1}$, I_{2k-1} as defined above. Then the zeros of $F_{2k-1}(z_\theta)$ and $F_{2k+1}(z_\theta)$ in $[\pi/10, 9\pi/10]$ are restricted to I_{2k-1} and I_{2k+1} respectively (in fact each $I_{j,2k-1}$ and $I_{j,2k+1}$ in $[\pi/10, 9\pi/10]$ contains an odd number of zeros). Moreover, the intervals are pairwise disjoint, and the $I_{j,2k-1}$ are separated by the $I_{j,2k+1}$.*

In addition to results about the location and interlacing properties of zeros, we also include an application to L -series which echoes (3.2).

Theorem 5.3 ([GLST] Theorem 4.1). *Let $k \geq 0$ be an integer and $E_{2k+1,\chi}(z)$ as in (5.1). Then*

$$\frac{2(-1)^k}{(2k)!L(2k+1,\chi)} \left(\frac{\pi}{2}\right)^{2k+1} = 4(2k+1) - 16 \sum_{\tau \in \mathbb{H} \setminus \Gamma(2)} \text{ord}_\tau(E_{2k+1,\chi}) \frac{1}{\lambda(\tau)}.$$

Similarly, we also have the following.

Corollary 5.4 ([GLST] Corollary 4.2). *Let $k \geq 0$ be an integer. Then*

$$\frac{2(-1)^k}{(2k)!L(2k+1,\chi)} \left(\frac{\pi}{2}\right)^{2k+1} = 4(2k+1) - 16 \sum_{\tau \in \mathbb{H} \setminus \Gamma(2)} \text{ord}_\tau(G_{2k+1,\chi}) \frac{1}{\lambda(\gamma_0\tau)}.$$

Similar results to Theorem 5.3 and Corollary 5.4 for even weight Eisenstein series on $\Gamma(2)$ can be found in [GLST].

It is clear that F. K. C. Rankin and Swinnerton-Dyer's beautiful result on the location of zeros of classical Eisenstein series has implications for many other classes of modular forms, including certain cusp forms and weakly holomorphic modular forms on $\mathrm{SL}_2(\mathbb{Z})$, as well as Eisenstein series for $\Gamma(2)$, certain Fricke groups, and Fuchsian groups of genus zero in general. However, the literature thus far considers only modular forms on groups of genus zero, and we see in the work of Shigezumi [Shi07] that even in genus zero, a more complex fundamental domain complicates the argument. In contrast, if p is prime, the genus of $\Gamma_0(p)$ grows with p , but the dimension of the Eisenstein space for $\Gamma_0(p)$ is only two, with $E_k(z)$ and $E_k(pz)$ forming a basis. The zero locations are known for these two series from the classical case. It would be very interesting to see if a general result exists for any genus.

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