Why is Ampère's law so hard? A look at middle-division physics

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Because mathematicians and physicists think differently about mathematics, they have different goals for their courses and teach different ways of thinking about the material. As a consequence, there are a number of capabilities that physics majors need in order to be successful that might not be addressed by any traditional course. The result is that the total cognitive load is too high for many students at the transition from the calculus and introductory physics sequences to upper-division courses for physics majors. We illustrate typical student difficulties in the context of an Ampère's law problem.

I. INTRODUCTION

We have all seen it happen. A student who got straight As in lower-division math and physics classes starts the post-introductory courses for physics majors and is totally bewildered to be suddenly getting Bs and Cs. The middle of the pack become angry or frustrated with the level of difficulty in our courses. "I just don't know how to get started!" echoes in the hallways. And too many of the weakest students give up or quietly disappear. Students who come to our office hours for help seem able to do the homework problems with a few hints, but freeze completely on exams. What is happening? To build more effective curricula we need to develop a better understanding of what makes the transition to upper-division physics so hard for some of our majors.

At some schools, as is the case at Oregon State University (OSU), the transition occurs in "middle-division" courses whose content is electrostatics and magnetostatics. The middle-division consists of those courses taken immediately after introductory calculus, introductory physics, and modern physics, and which serve to introduce the major. At other schools the middle-division courses cover topics such as waves, mathematical methods, or classical mechanics. For the past nine years, we have been focusing on this transition in two NSF-funded projects at OSU. In this paper, we share the insights we have gained that are relevant to the teaching of these courses.

The Paradigms in Physics program[1] comprises a complete reorganization and revision of upper-division theory courses to cultivate students' analytical and problem-solving skills. The nature and goals of the program as a whole have been discussed in detail.[2,3] One of the goals in the first few courses is to ease the problems that students have transitioning from lower-division to upper-division courses. Group activities require students to employ geometric reasoning and build mathematical skills in the context of strongly focused physical examples. We encourage movement away from routine problem-solving following well-defined templates and toward the use of multiple representations and synthesis.

The purpose of the Vector Calculus Bridge Project[4] is to understand the differences in perspective between mathematicians and physicists and why these differences cause transition problems for students. Informed by these understandings we designed and classroom-tested curricular materials at OSU. We also developed resources for mathematics faculty to help them appreciate the needs of their physical science and engineering students. These resources include a series of papers[5–8] that emphasize the importance of the vector differential $d\vec{r}$ in both rectangular and curvilinear coordinates, group activities and an instructor's guide focused on student development of geometric reasoning, and an ongoing series of faculty development workshops.[9] The Bridge Project has now evolved to the point that we are using what we have learned to address the educational needs of students in middle-division courses.

The Paradigms and Bridge projects are perhaps unique in terms of the sheer scope of the curriculum that they address. From this broad perspective we have learned that there are overarching expectations that we implicitly hold for our students: students at this level are required to solve problems involving many steps and to engage in complex logical arguments; they must generalize their nascent conceptual understanding to examples that involve unexpected additional structure; and they must pull together resources from many previous experiences, recognizing that what they learn today is not simply related to what they learned yesterday, but may involve a web of information from many previous courses — learning is not linear.

The expectations on this abstract list should come as no surprise. How do they impact our students in practice? To make our discussion concrete, we include a detailed task analysis of an Ampère's law problem, highlighting common student difficulties. None of the individual difficulties will sound overwhelming; once students have had a chance to address them, they find the solutions straightforward. Nevertheless, so many ideas come together that, even under the best of circumstances, many students need to scramble to keep up. Our task analysis suggests the question, "Is the total cognitive load in middle-division courses too high?" Synthesis has become so automatic to us that we may fail to recognize how new it is for our students. Are we giving them sufficient resources to be able to do everything we ask of them?

In Sec. II we give a broad discussion of two major differences between the way mathematicians and physicists use mathematics. In the rest of the paper we explore the consequences for students as they try to bridge this gap by applying what they have learned in mathematics courses to physics courses beyond the introductory level. In Sec. III we introduce a standard Ampère's law problem and discuss typical textbook solutions. Section IV discusses a detailed task analysis of this problem. In Sec. V we return to the broader theme of the capabilities that we want our middle-division students to be constructing, and suggest that designing curricula that pay explicit attention to the transition students need to make may help more students be successful. Section VI briefly links our work to the work of others.

This paper does not pretend to report on education research (but see Ref. 10). We have not done careful studies to learn how prevalent particular student problems are. Nor have we systematically compared the results of educational interventions that we suggest here to either traditional methods or those based on education research. It would be impossibly cumbersome for us to write, or the reader to read, properly qualified sentences; we ask the reader's indulgence. When we write, "Students think . . . ," we really mean, "In our many years of working with students, faculty, and TAs from a diverse set of institutions, we suspect that at least some, and probably a significant number of students may think . . . , and that regardless of what they are actually thinking, if we tailor our educational interactions with them as if they think . . . , then apparently, it seems to help them learn more and/or they at least appear to be more "satisfied" with their learning experience, without our actually assessing that."

In all seriousness, we hope that what we suggest will not only provide numerous fruitful questions for education research but also inspire traditional educators to look more closely at what is happening in their classrooms.

II. MATHEMATICS IS NOT PHYSICS

Mathematicians are responsible for much of the lower-division education of our students, and yet mathematicians and physicists view mathematics in inherently different ways. This contrast in perspective has dramatic repercussions when our students try to apply the mathematics they have learned in the physics classroom, as illustrated in Sec. IV. We have found that many of the differences between the problem solving strategies of mathematicians and physicists fit under two main headings.

A. Physics is about things

In our conversations with physics and mathematics faculty the most striking differences arise from the fact that physics is about describing fundamental relationships between physical quantities whereas mathematics is about rigorously pursuing the consequences of sets of basic assumptions. Conventional lower division mathematics is primarily about teaching students to manipulate mathematical symbols according to well-defined rules without asking about the interpretation of these symbols. Calculus reform has helped somewhat, but even application-based curricula that are designed to stress multiple representations have limited time to focus on the interpretation of results. Rightly, interpretation is the realm of science. As professionals who have spent our careers interpreting equations and finding ways of representing information, the first question that we ask about a new formula is, "What physical quantities do the various symbols represent?" Our eyes are trained to pick out the constants and variables and we automatically recognize those quantities that increase or decrease as other variables change. We ask ourselves if the relations we see are the ones we expect based on our experience with simpler examples. It can be difficult to remember that these are new questions and ways of thinking for our students.

B. Physicists can't change the problem

Because mathematics is abstract, it is possible to design courses, at least at the K–14 level, that focus on a single problem-solving method at a time. Professional physicists do not have this luxury. The first time that students are asked to combine many different ideas and problem-solving strategies to obtain a final answer may be in middle-division physics. Our students have already memorized many facts, grappled with a number of concepts, and have a toolbox containing many independent skills. What they now need is a foundation for their learning in physics and their future ability to solve new problems. This foundation consists not so much of learning *how* to solve new kinds of problems as of *connecting* the knowledge they already have into a coherent understanding of what it means to solve problems.

III. THE AMPÈRE'S LAW PROBLEM

Ampère's law problems are a common stumbling block for many students in middle-division E&M courses. An analysis of such a problem serves as an excellent example for exploring the challenges students face as they make the transition from introductory courses to courses in the major.

Ampère's law for magnetostatics states that the line integral around a closed loop of a physically realizable magnetic field is equal to a dimensionful constant times the total current enclosed by the loop:

$$\oint_{\text{any closed loop}} \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{r}} = \mu_0 I_{\text{enc}}. \tag{1}$$

For special cases of high symmetry this law is used to find the value of the magnetic field due to a steady current.

Consider the following typical Ampère's law problem taken directly from our favorite upper-division E & M text:[11]

> A steady current I flows down a long cylindrical wire of radius a. Find the magnetic field, both inside and outside the wire if the current is distributed in such a way that J is proportional to s, the distance from the axis.

We have chosen to discuss this problem because the geometry of magnetostatics is trickier than the geometry of electrostatics. Many of the issues that we discuss are common to earlier electrostatics problems, and indeed a well-structured curriculum would begin to address them there. For some students this second experience with Ampère's law actually clarifies similar problems involving Gauss's law. Subsequent electromagnetic theory topics must, in turn, build on a firm foundation of both electrostatics and magnetostatics.

This would be an excellent time for the reader to pause and attempt to solve the problem as we will be discussing tricky parts of the solution in some detail. If you choose not to bother, you might want to think about how often your students also choose not to work through an example. What are the implications for your pedagogical strategies?

A. The Usual Solution

Before we begin our discussion of the plethora of challenges students face with Ampère's Law problems, it is illuminating to consider the amount of explanation typically given to problems of this type. We quote the entire solution given in the same standard textbook to the (somewhat simpler) problem of finding "the magnetic field a distance s from a long straight wire carrying a steady current I."[12]

> *Solution.* We know the direction of $\vec{\boldsymbol{B}}$ is "circumferential," circling around the wire as indicated by the right-hand rule. By symmetry, the magnitude of $\vec{\boldsymbol{B}}$ is constant around an Ampèrian loop of radius s, centered on the wire. So Ampère's law gives
>
> $$\oint \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{l}} = B \int dl = B 2\pi s = \mu_0 I_{\text{enc}} = \mu_0 I, \tag{2}$$
>
> or
>
> $$B = \frac{\mu_0 I}{2\pi s}. \tag{3}$$

Solutions to subsequent more complex examples spend some time discussing the details of the symmetry arguments needed to determine the direction of the magnetic field, but no further details on any other aspects of the problem.

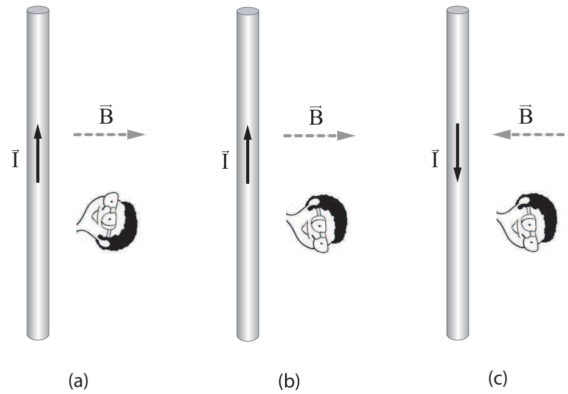

FIG. 1: The symmetry argument used to argue that the magnetic field due to an infinite straight wire has no radial component: Assume such a component exists (a). Facing the other way (b) and reversing the current (c) does not change the physics, but would reverse this component.

B. The Overall Strategy

The overall strategy in such problems is to choose an Ampèrian loop that is parallel to the magnetic field at every point and for which the magnitude of the magnetic field is constant. Then the magnitude of the magnetic field can be pulled out of the integral and Ampère's law becomes a simple statement about the magnitude of the field. This strategy can only be used to find magnetic fields only in physical settings with an extraordinarily high degree of symmetry — infinitely long, straight, round wires of various thicknesses with current densities that depend only on the distance from the axis, infinite planes of various thicknesses with planar current densities, solenoids and toroids.

It is disarmingly easy to turn such problems into templates. If students only see problems where the required symmetry exists, they can get the right answer without any conceptual understanding at all, especially if the appropriate Ampèrian loop is specified in the problem. To evaluate the left-hand side of Eq. (1), all you need to do is parrot the words "due to symmetry," pull B out of the integral, and multiply by the length of the loop. For square loops, say "oops" when you are told not to include the parts of the loop that are perpendicular to the current. For the right-hand side, most problems deal with constant current densities, so multiply this density by the cross-sectional area of the nearest geometric object in sight. Finally, simplify the resulting equation and solve for B.

IV. TASK ANALYSIS

We now turn to a task analysis of the Ampère's law problem given in Sec. III. We remind the reader that this problem involves a cylindrical wire of finite thickness. The problem-solving tasks we discuss fall naturally into two categories that reflect emerging student skills: "Geometry and symmetry" — those tasks that require students to draw pictures and to think about the relationships of algebraic symbols to objects in physical space, and "What sort of a beast is it?" — those tasks that require students to understand the physical attributes of the quantities associated with algebraic symbols.

A. Geometry and Symmetry

$\vec{B}$ *varies in both magnitude and direction.* It helps right at the beginning of an Ampère's law problem to know both the direction of the magnetic field and the variables on which the magnitude depends. For Gauss's law the analysis appears to be simpler. The electric field typically decreases with distance from the source (charge) and the field also usually points away, so that students are able, without penalty, to mush together these two facts in their minds. Magnetic fields also typically decrease with distance from the source (current), but the direction is . . . which way?

Symmetry. It is obvious "by symmetry" that the magnetic field will point "around" the wire and have a magnitude that depends only on r, the distance from the axis. We have all made this argument so many times that we no longer realize how subtle it is.

Part of the argument is straightforward. An observer who moves from one point to another, either circumferentially around the wire or parallel to the wire, cannot tell that anything has changed. Therefore, the magnitude of the magnetic field must depend on r alone. But what about the direction? For an infinitely thin, straight wire, students are tempted to reason the direction using the right-hand rule, an argument that assumes part of what they are asked to prove. When they consider, as in our case, the magnetic field due to the individual parts of the current inside a wire with finite radius R, the argument is not so simple. Students and faculty alike can get themselves tied up in knots trying to argue which components add and which cancel. A nicer argument[13] assumes that the magnetic field has an outward-pointing radial component and considers an observer initially facing in the direction of the current, as shown in Fig. 1(a). If the observer turns around in place, as shown in Fig. 1(b), the action does nothing to the direction of the magnetic field. Now reverse the current as shown in Fig. 1(c) so that the reversed observer again faces in the direction of the current. The observer expects to see the same magnetic field as at the beginning because the "universe" appears as it did in the beginning. But the magnetic field depends linearly on the current: reversing the direction of the current makes the assumed outward pointing magnetic field reverse and point in.[14] A contradiction.

For several years we have modeled this type of argument in lecture using appropriate props. Immediately thereafter, students are asked to solve a similar problem in small groups. Almost all are unsuccessful. This kind of reasoning, assuming that something might be true and then following this idea to its logical conclusion, is common in physics. However, formal proof-by-contradiction is no longer routinely taught in high school mathematics classes — a prime example of content that does not appear in any traditional course.

Arguing away the component parallel to the wire is not so easy. The physical argument that the magnetic field must fall off at infinity is plausible, but fails for an infinite sheet of current. The only way we know to establish the lack of a parallel component is to use the Biot-Savart law, whose cross product forces the magnetic field to be perpendicular to the current. This argument is compelling, but far from obvious; the Biot-Savart law is not yet part of students' instinctive knowledge of physics.

Choosing an Ampèrian loop that isn't there. The statement of Ampère's law in Eq. (1) informs students that they must integrate over some closed loop. Which one? There are several pitfalls. First, some students will look around for a curve that already exists: for an infinitesimally thin, straight wire, they sometimes choose the wire itself; for an infinitesimally thin, circular wire they almost always choose the wire itself; for the thicker straight wire that we consider here, some choose a circle around the wire, lying precisely in its surface. It is difficult for many students to grasp the need to choose an imaginary loop.

Which loop should they choose? A circular loop around the wire. What radius should it have? They have to choose every possible radius, one at a time. When students first choose a loop, it is important for them to think of the radius r as a constant. After the integration, r becomes a parameter. To understand the range of values that r can take ($0 \leq r < a$ or $a < r < \infty$), students must also recognize that r represents a geometric coordinate. Students who are used to problems in lower-division courses for which the solutions are numbers with units, blur the differences between constants, variables, and parameters. We like to ask students to identify the constants and variables in the general linear equation $ax + by + c = 0$. Short of the statement that the constants are from the beginning of the alphabet and the variables are from the end, the best answer is that the variables are the symbols whose values change and the constants are the symbols whose values do not change ... until they do!

Furthermore, it is not obvious to the novice that a horizontal loop is the "obvious" choice. We suspect that simple memory aids the expert. You already have to know the result you are trying to show, namely the direction of the magnetic field, to make this choice. Loops with segments parallel to the wire could yield information about other components of the magnetic field. Two of us spent a delightful hour on a long car trip trying to determine how much of the magnetic field can be found by exploiting a variety of Ampèrian loops alone, without making any explicit symmetry arguments for the direction. It's a wonderful exercise in geometry and logic; we encourage the interested reader to try it. And we encourage all readers to recognize that it is the rare student who would find delight in the exercise.

Inverse problems. In our problem, the use of Ampère's law is at heart an inverse problem; the desired information cannot simply be obtained by solving Eq. (1) algebraically for $\vec{B}$. The magnetic field that the students need to solve for lies inside both an integral and a dot product. How do they get the magnetic field out of a box with that much wrapping around it? Most students just yank. How do you convince them that it's not so simple?

Knowing whether or not you can get $\vec{B}$ out of the box requires you to know about the geometric nature of the wrapping. The first clue is to recognize that integrals are sums, not necessarily areas. The common first year calculus mantra that "integrals are areas" can be very misleading to students in a multivariable setting. In electromagnetic theory students need to imagine chopping up a part of space, calculate some physical quantity on each of the individual pieces, and add up the physical quantities from the individual pieces to obtain the total value of the physical quantity. On the left-hand side of Eq. (1) the Ampèrian loop is chosen so that the magnetic field is constant. Then the separate pieces of magnetic field (times an infinitesimal length) that the students are adding are identical.

But what does it mean for the magnetic field to be constant? What is that dot product doing there? The dot product is a geometric operator that projects vectors onto other vectors. Its role in Eq. (1) is to find the component of the magnetic field vector parallel to the Ampèrian loop. Many students think of the dot product only in terms of its algebraic formula in rectangular components. (See Ref. 15 for a fuller discussion of this issue.) When the dot product becomes troublesome to think about, it tends to disappear, without a warning of its passing. Unfortunately, because vanishing the dot product and unceremoniously yanking the magnetic field out of the integral give the right answer, it can be difficult to notice how cavalier some students are. Watch for it!

Curvilinear coordinates. E & M is more about spheres and cylinders than it is about planes or more esoteric shapes. In multivariable calculus courses a standard surface is the paraboloid, which is not typically encountered in physics problems. The advantages of using curvilinear coordinates when doing integrals over such surfaces is an example of content that is not sufficiently owned by either traditional mathematics courses or traditional physics courses. Our problem would be much more difficult to do in rectangular coordinates. The need to explicitly choose a coordinate system is not automatic to some of our students.

When mathematics faculty teach cylindrical and spherical coordinates, they do not use the same language as physicists. For instance, they rarely discuss the basis vectors such as $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\phi}}$ that are adapted to these coordinates; indeed, many have never even heard of these geometric objects. Thus, when a student is first told that the magnetic field around a current carrying wire points in the $\hat{\phi}$ direction, their first response is likely to be, "phi hat?" See Ref. 16 for a more detailed discussion of this issue.

B. "What sort of a beast is it?"

Because students do not have much experience thinking of mathematics as representing physical things, they may not automatically ask themselves questions about a particular symbol. What physical quantity does it represent? Is it a vector or a scalar? What dimensions does it have? Is it finite or infinitesimal? Is it a variable, a constant, or a parameter? To prompt students to ask themselves these kinds of questions, we often ask them "What sort of a beast is it?"

What is a steady current? Magnetostatics is a curious subject. Currents are created by moving charges. If the charges are moving, what is static? We ask the students to pretend that they are charges and to move around the room randomly. Then we ask the students to move so that an imaginary "magnetic field meter" held by the teacher will read a magnetic field that is constant in both magnitude and direction. It takes only a few seconds for the students to figure it out. But lots of mental light bulbs go on in those few seconds. This is a *steady* current!

What is density? If you ask what density is, students at this stage will typically answer either "mass divided by volume" or "grams per centimeter cubed" (or an equivalent statement in another set of units). These responses are very interesting in terms of what they tell us about students' conceptual understanding. The first response indicates that the students tend to think of concepts in terms of formulas that allow them to calculate the answer to a problem. Listen to them talk to each other. The second response indicates that they equate the physical thing with the units used to measure it. Although each of these answers contains a necessary idea, there are several ways in which students' understandings need to be generalized for them to be able to solve our Ampère's law problem. Mass is the earliest physical quantity for which students use the word density. Somewhat later they learn to use the word density for charge. For Ampère's law they need to consider current density, which for most students is a totally new use of the word.

Densities can vary from place to place. In most of their previous schooling, students consider only global quantities rather than local ones: densities are constant rather than variable. After all, mass densities usually are constant — what is the density of ice or lead? Mass densities may, for example, change slightly with temperature, but not typically from place to place. Even more germane is that less advanced students have a mathematical limitation. There is not much point assigning problems about densities that vary from place to place before students have studied calculus because they cannot use a variable mass density to find the total mass until they can integrate and they cannot explore a variable mass density until they can differentiate. It is illuminating to look up the sections on density in a typical calculus text; several different applications from the physical and social sciences will all be run together in a single section. Even when the book has a complete description of each concept, it will probably also provide a formula that students can use for template problem solving without reading the description or wrestling with the concept. It is unrealistic to expect a fast-paced calculus course to spend much time teaching the context of applications as well, so the most likely scenario is that your students have solved at most one or two variable density problems. We have seen a number of students who, when asked to calculate the mass of a planet with mass density $\rho = kr^2$, simply take the expression for ρ and multiply it by the volume of a sphere. After all, total mass equals density times volume, doesn't it?

Line, surface, and volume densities are all different. In our Ampère's law problem, the current is distributed through the entire volume of a wire. In similar problems current might be considered to flow only along a surface or through an infinitely thin wire. Such problems are special cases of a volume current, where the distribution of the current in one or more dimensions is so constrained that these dimensions are idealized away. Students' greatest level of classroom experience is with line currents, the most idealized case.

Pedagogically, we can imagine two ways to handle the differences among these densities in the classroom. One is to define the different types of densities as different physical quantities, with different units, that require differing numbers of integrals to find the total value of the current. The other is to use these differences as an opportunity to exploit the sophisticated mathematics of theta and delta functions and explicitly discuss surface and line currents as limiting cases of volume currents. The first way causes the least disruption in the students' attention to the central question of Ampère's law. The second way seems to be the most satisfying to students who are trying to develop an understanding of current.

Total current is a flux. By the time they get to Ampère's law, students have typically encountered both mass and charge densities. Students expect a density to have dimensions of the total physical quantity divided by the geometric quantity that describes the type of density (line, surface, or volume). Line charge densities are coulombs per unit length, surface charge densities are coulombs per unit area, and volume charge densities are coulombs per unit volume. So, simple pattern matching would indicate that volume current density is current per unit volume. Right? Wrong. Volume current density is current per unit area. What happened? By the pattern matching argument, the current density should have units of $\mathrm{Q/T\,L^3}$. To obtain the total current, we thus expect to have to integrate the current density over a volume. But this reasoning is not correct.

Although total charge is found by chopping up a line, surface, or volume, and adding up the charge on each piece, total current is found by setting up a gate and finding out how much charge passes through the gate in unit time. We therefore obtain the total current by finding the flux of the given current density across the cross-section, that is,

$$I = \int \vec{\boldsymbol{J}} \cdot d\vec{\boldsymbol{A}}. \tag{4}$$

Linear current density refers to current along a one-dimensional curve. The appropriate gate is just a single point and the total current at this point is identical to the linear current density at this point with dimensions $\mathrm{Q/T}$. The term surface (volume) current density refers to current spread out along a two- (three-) dimensional part of space and the gate is a one- (two-) dimensional cross section. The total current is found by taking a one- (two-) dimensional flux integral over that cross section.

If the students are already moving around the room (as described earlier to demonstrate a steady current), it can be very helpful to put up a "gate" and ask them how many charges (people) will pass through the gate in the next second. The fact that current density is just the expected type of charge density times the velocity seems to resolve the issue of dimensions for many students.

V. SUMMARY

Students need all of the following capabilities to solve our Ampère's law problem: the ability to (1) recognize and use symmetry arguments, (2) represent physical quantities symbolically and keep track of their properties, (3) move smoothly between various representations, (4) make geometric arguments such as interpreting integrals as sums, and (5) recognize and solve subtle inverse problems. All of these capabilities are common to any middle-division course. Although all of these skills are essential, it is rare to see them explicitly listed as course goals in these transitional courses. Without explicit recognition, they are destined to take a back seat to traditional content goals.

Given all of the difficulties that students have, one might reasonably ask why we even have students do these problems. The technique works only for a few cases with an unphysically high degree of symmetry. The problems seem easy but are actually hard. What is the point? The point is that you have to be able to think like a physicist to do these problems. You have to understand something about the physical meaning of the quantities involved. You have to know what geometric properties things have. You have to pull together lots of different content. Once you are done, if you look at the physical and geometric meaning of your answer, it tells you a lot about the behavior of magnetic fields in certain special geometries. Since magnetic fields add linearly, these special cases become the building blocks for more complex cases. And finally, the answers are a lovely opportunity to talk about idealizations and limiting cases, finite lengths and edge effects, and many other physical explorations. This is the very stuff of which theoretical physics is made.

Learning how to be a physicist is far more difficult than we realize. It involves change in the students' understanding of what it means to solve problems. We can make problems at this level easier for the students to solve by turning them into templates in various ways, but, when we do, we risk short circuiting the transformation process. If we value the transformation itself, it is important that we recognize how much we are asking of the students. If we want to support this change, we must break up learning what it means to solve problems, rather than problem solving, into steps. Our experience shows that when this is done, the vast majority of students are capable of making the transformation.

VI. OTHER RESOURCES

If you assume students have a particular skill, it helps to ask yourself where they might reasonably have learned it. Check! We have been stunned any number of times. Sometimes, just knowing that the students have not seen something allows you to address it easily.

In both the Paradigms and Bridge Projects we have designed our curricula to take responsibility for helping students develop the capabilities that we have discussed here. On our websites,[1,4] you can find sample syllabi that pay explicit attention to the development of students' understanding of problem solving, not just content. The courses on Symmetries & Idealizations and Static Vector Fields are particularly relevant to Ampère's law. You can also find many activities, instructor's materials, and information about faculty development workshops. We are also building a website[17] based on rich descriptions of individual activities. We would be happy to hear from those who are interested in building a community to investigate these ideas.

Our understanding of student difficulties with Ampère's law problems builds on a long heritage of education research from sweeping theoretical treatises to practical research-based curricula. For the traditional research physicist or mathematician with little or no education research background, the task of entering this vast literature can be daunting. Here are a few brief guideposts.

1. Physics educators have investigated student difficulties in electricity and magnetism and developed new curricula for teaching E & M at the introductory level. An excellent resource for this work is the Resource Letter by McDermott and Redish,[18] with its extensive annotated bibliography. More specifically, Maloney, Hieggelke and colleagues have begun to address the evaluation of student conceptual understanding.[20] In addition, they have recently published a collection of classroom tasks designed to help students develop a better conceptual understanding of electricity and magnetism.[21] Although these studies have focused primarily on conceptual understanding at the introductory level, they serve as an excellent resource for gaining a better understanding of our students.

2. A delightful, readable introduction to teaching physics by Redish also serves as an overview to the current status of physics education research and contains an excellent bibliography of more recent PER references.[19]

3. It is interesting to speculate on the extent to which synthesis at the middle division can be scaffolded by lower-division curricula that explicitly emphasize problem-solving. Some examples of such lower-division curricula are the Context Rich Problems of the University of Minnesota group[22] and the introductory text by Chabay and Sherwood.[23]

4. We are inspired by Vygotsky's admonishment[24–26] to design our curriculum to keep the level of the content as much as possible in the "zone of proximal development," that magic region of instructional space between what the students are able to learn without our help and what they are not able to learn, even with our help.

5. Krutetski[27] distinguishes three types of student reasoning: analytic, geometric, and harmonic. We are intrigued that students at this level are not demonstrably harmonic, that is, in problem-solving interviews most do not spontaneously move back and forth between analytic and geometric reasoning.[10]

6. The research area of multiple representations acknowledges that students must be able to exploit several different representations of mathematical or physical quantities to be good problem solvers. Early resources from mathematics education research can be found in Janvier.[28] This work informed the calculus reform movement and is clearly expressed in the "Rule of Three (or Four)."[29] Early resources from physics can be found in Heuvelen.[30]

7. A good entry point to the literature on cognitive load is the paper by Sweller.[31]

8. The field of expert-novice problem solving involves studies of the differences between the way experts solve problems and the way novices solve the same problems. The challenges we describe as students pass between the lower division and the upper division is in essence a part of the transition from novice to expert problem-solvers. Important early papers include Refs. 32 and 33.

9. The transfer problem, namely how students learn to use ideas, information, or skills acquired in one setting in another setting, is a central and longstanding field of education research. A new book contains articles and references from several disciplines.[34]

10. An article on mathematical problem solving by Schoenfeld[35] provides a rich introduction to the mathematics education literature and an extensive bibliography.

11. It is dismaying that many physics majors at the middle-division level might still be having difficulties with proportional reasoning,[36] but a lack of fluency in this area may well underlie the student problems with density that we have discussed. An interesting paper by Kanim[37] in the context of charge density suggests that students still have such problems late in the lower division.

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