Nonlinear Analysis: Real World Applications 54 (2020) 103110

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

www.elsevier.com/locate/nonrwa

Persistence and extinction of nonlocal dispersal evolution equations in moving habitats

Patrick De Leenheer [a], Wenxian Shen [b], Aijun Zhang [c,*]

[a] Department of Mathematics, Oregon State University, Corvallis, OR 97331, United States of America
[b] Department of Mathematics and Statistics, Auburn University, AL 36849, United States of America
[c] Department of Mathematics, University of Louisiana, Lafayette, LA, 70504, United States of America

ARTICLE INFO

Article history:
Received 19 August 2019
Received in revised form 21 January 2020
Accepted 22 January 2020
Available online 8 February 2020

Keywords:
Nonlocal dispersal
Moving habitats
Traveling wave
Persistence
Extinction

ABSTRACT

This paper is devoted to the study of persistence and extinction of a species modeled by nonlocal dispersal evolution equations in moving habitats with moving speed c. It is shown that the species becomes extinct if the moving speed c is larger than the so called spreading speed c^*, where c^* is determined by the maximum linearized growth rate function. If the moving speed c is smaller than c^*, it is shown that the persistence of the species depends on the patch size of the habitat, namely, the species persists if the patch size is greater than some number L^* and in this case, there is a traveling wave solution with speed c, and it becomes extinct if the patch size is smaller than L^*.

1. Introduction

Nonlocal dispersal equations have been widely employed as models in the applied fields such as biology, material science, neuroscience, chemistry and ecology [1–7].

The current paper is to investigate the following nonlocal dispersal equation,

$$\frac{\partial u(t,x)}{\partial t} = \int_{\mathbb{R}} k(y-x)u(t,y)dy - u(t,x) + f(x-ct,u)u(t,x), \quad x \in \mathbb{R}. \tag{1.1}$$

We assume that $k(\cdot) : \mathbb{R} \to \mathbb{R}^+$ is a C^1 convolution kernel function that satisfies the following:

(H1) *$k(\cdot) \in C^1(\mathbb{R}, [0,\infty))$, $k(z) = k(-z)$, $\int_{\mathbb{R}} k(z)dz = 1$, $k(0) > 0$, and there exist $\mu, M > 0$ such that $k(z) < e^{-\mu|z|}$ and $|k'(z)| < e^{-\mu|z|}$ for $|z| > M$.*

Typical examples satisfying (H1) include the probability density function of the normal distribution $k(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ and any C^1 symmetric convolution kernel functions supported on a bounded interval.

* Corresponding author.
E-mail address: zhangai@tigermail.auburn.edu (A. Zhang).

https://doi.org/10.1016/j.nonrwa.2020.103110
1468-1218/

Biologically, in (1.1), the term $\int_{\mathbb{R}} k(y-x)u(t,y)dy - u(t,x)$ characterizes the dispersal of the organisms that exhibits long range internal interactions. $f(x-ct,u)$ is the reaction term that is related to the growth of species. Noting a speed c in the reaction term $f(x-ct,u)$, biologically we assume the reaction of the populations will change with the moving habitat of speed c due to some external environment change, like climate change. Without loss of generality, we assume that $c \geq 0$. If $c < 0$, biologically it means that the habitat moves in an opposite direction. Mathematically, by changing variables with $\tilde{c} = -c$ and $\tilde{x} = -x$, we can obtain an equivalent equation as (1.1) for $\tilde{c} > 0$ and $\tilde{x}$. Let $\phi_{\pm}(x)$ be C^1 functions satisfying that $\phi_{\pm}(\pm x) = 1$ for $x \leq 0$, $\phi_{\pm}(\pm x) = 0$ for $x \geq 1$, $\phi_{+}(x) = \phi_{-}(-x)$, $\phi'_{+}(x) \leq 0$ and $\phi'_{-}(x) \geq 0$ for $x \in \mathbb{R}$. We assume that f satisfies

(H2) *There are $r, q, L, L_0 > 0$ such that $f(x,u)$ is C^1 in (x,u); $f(x,u) = -q$ for $|x| \geq L + L_0$; $f(x,u) = r(1-u)$ for $|x| \leq L$; $f(x,u) = -q + (r(1-u)+q)\phi_{+}\big(\frac{x-L}{L_0}\big)$ for $L < x < L + L_0$; and $f(x,u) = -q + (r(1-u)+q)\phi_{-}\big(\frac{x+L}{L_0}\big)$ for $-L - L_0 < x < -L$.*

Observe that $f(x,\cdot) = f(-x,\cdot)$ and

$$\lim_{L_0 \to 0^+} f(x,u) = \begin{cases} -q & \text{for} \quad x > L \\ r(1-u) & \text{for} \quad -L \leq x \leq L \\ -q & \text{for} \quad x < -L. \end{cases}$$

Here is an example of $f(x,u)$ which satisfies (H2),

$$f(x,u) = \begin{cases} -q & \text{for} \quad x \geq L_0 + L \\ -q + \frac{r(1-u)+q}{2}\big(1 + \cos\frac{\pi(x-L)}{L_0}\big) & \text{for} \quad L < x < L_0 + L \\ r(1-u) & \text{for} \quad -L \leq x \leq L \\ -q + \frac{r(1-u)+q}{2}\big(1 + \sin\frac{\pi(2x+2L+L_0)}{2L_0}\big) & \text{for} \quad -L_0 - L < x < -L \\ -q & \text{for} \quad x \leq -L_0 - L. \end{cases}$$

Assumption (H2) indicates that the region $|x| \leq L$ is the favorable habitat for the species; there is a finite mortality rate q outside of the region $|x| \leq L + L_0$; and the region $L \leq |x| \leq L + L_0$ is the transition region. It should be pointed out that the specific form of $f(x,u)$ in the transition region in (H2) is just for concreteness. It can be replaced by a general form in this region as long as $f(x,u)$ is C^1 in (x,u), is non-increasing in u for $u \geq 0$, and is non-decreasing with respect to L.

Recently, modeling the effects of global climate change on populations has drawn a lot research attention in the scientific community [8–12]. Berestycki et al. in [8] and Li et al. in [10] considered the following reaction–diffusion equation, for $c > 0$

$$\frac{\partial u(t,x)}{\partial t} = D\frac{\partial^2 u(t,x)}{\partial x^2} + f(x-ct,u)u(t,x), \quad x \in \mathbb{R}. \tag{1.2}$$

but with different reaction term f. In [8], f is assumed to be

$$f(x-ct,u) = \begin{cases} r-u & \text{for} \quad |x-ct| \leq L \\ -q & \text{otherwise} \end{cases} \tag{1.3}$$

for some $L > 0$, which indicates that favorable habitat is bounded and surrounded by unfavorable habitat. It should be pointed out that the paper [11] addresses the same question as in [8], but focuses on the effect of a moving climate on the outcome of competitive interactions between two species, and that it is in [11] that terms of the form $f(x-ct,u)$ appear for the first time. It should also be pointed out that the [9] gives a comprehensive study of the problem in terms of integrodifference equations in the presence of climate change. One is also referred to [12] for a study on the critical speed for extinction, and the role that the dispersal and growth play in persistence in integrodifference equations with shifting species ranges. In [10], f is of the form

$$f(x-ct,u)=r(x-ct)-u, \tag{1.4}$$

where r is continuous, non-decreasing and bounded with $r(-\infty)<0$ and $r(\infty)>0$. Note that, in this case, the favorable habitat is unbounded. The reader is also referred to [13] and [14] for the study of lattice differential equations and for nonlocal dispersal equations with the nonlinear reaction term (1.4).

Interesting dynamical issues for (1.1) and (1.2) include the persistence and extinction of the population, in particular, the dependence of the persistence and extinction on the speed c and the patch size of the moving habitat. It will be seen that the persistence and existence of traveling wave solutions with speed c are strongly related, and the extinction and nonexistence of traveling wave solutions with speed c are also strongly related. In the current paper, we are interested in the existence and nonexistence of traveling wave solutions of (1.1) with speed c, i.e., positive solutions of the form $u(t,x)=v(x-ct)$.

To this end, we consider solutions of (1.1) of the form $u(t,x)=v(t,x-ct)$ with $v(t,x)$ being differentiable. Then letting $\xi=x-ct$, $v(t,\xi)$ satisfies

$$\frac{\partial v(t,\xi)}{\partial t}=c\frac{\partial v(t,\xi)}{\partial \xi}+\int_{\mathbb{R}}\kappa(\eta-\xi)v(t,\eta)d\eta-v(t,\xi)+f(\xi,v)v(t,\xi),\quad \xi\in\mathbb{R}. \tag{1.5}$$

We remark that Eq. (1.5) models the nonlocal dispersal, advection and reaction of a single species in a heterogeneous environment. The number c measures the advection velocity. The term $c\frac{\partial v(t,\xi)}{\partial \xi}$ describes the drift of the population with the constant speed c. Advective processes occur, for example, in a river or ocean, where organisms may drift, sink or rise due to the water flows and their own relative weights compared with the surrounding medium (i.e water).

Note that any nontrivial stationary solution $v(\xi)$ of (1.5) satisfies

$$cv'(\xi)+\int_{\mathbb{R}}\kappa(\eta-\xi)v(\eta)d\eta-v(\xi)+f(\xi,v)v(\xi)=0,\quad \xi\in\mathbb{R}, \tag{1.6}$$

and gives rise to a traveling wave solution $u(t,x)=v(x-ct)$ of (1.1).

Let

$$X=C^b_{\text{unif}}(\mathbb{R})=\{u\in C(\mathbb{R})\mid u\quad \text{is uniformly continuous and bounded on}\quad \mathbb{R}\}$$

with norm $\|u\|=\sup_{x\in\mathbb{R}}|u(x)|$, and

$$X^+=\{u\in X\mid u(x)\geq 0\}.$$

Consider initial value problem for (1.1) and (1.5) on X. By semigroup theory (See [15,16]), for any $u_0\in X$, (1.1) has a unique local classical solution $u(t,x;u_0)$ with $u(0,x;u_0)=u_0(x)$, and for any $u_0\in X$, and (1.5) has a unique local mild solution $v(t,\xi;u_0)$ with $v(0,\xi;u_0)=u_0(\xi)$. Moreover, if u_0 is differentiable and $u_0'(\cdot)\in X$, then $v(t,\xi;u_0)$ is the classical solution of (1.5).

We say that *persistence* occurs in (1.1) if for any $u_0\in X^+$ with $\inf_{x\in\mathbb{R}}u_0(\xi)>0$,

$$\liminf_{t\to\infty}\inf_{|\xi|\leq K}v(t,\xi;u_0)>0$$

for any $K>0$. We say that *extinction* occurs in (1.1) if for any $u_0\in X^+$,

$$\lim_{t\to\infty}\sup_{\xi\in\mathbb{R}}v(t,\xi;u_0)=0.$$

For $r>0$, let c^* be the spreading speed of

$$u_t=\int_{\mathbb{R}}k(y-x)u(t,y)dy-u(t,x)+r(1-u)u,\quad x\in\mathbb{R}, \tag{1.7}$$

that is,

$$c^* = \inf_{\mu>0} \frac{\int_{\mathbb{R}} e^{-\mu z} k(z)dz - 1 + r}{\mu} \tag{1.8}$$

(see Proposition 2.4).

For given $\lambda > -q$, let

$$g(\mu;\lambda) = c\mu + \int_{\mathbb{R}} e^{\mu\eta} k(\eta) d\eta - 1 - q - \lambda.$$

Then

$$g_{\mu\mu} = \int_{\mathbb{R}} \eta^2 e^{\mu\eta} k(\eta) d\eta > 0,$$

and $g(0;\lambda) = -q - \lambda < 0$. Note that $g(\mu;\lambda) \to \infty$ as $\mu \to \pm\infty$. Hence there are $\mu_-(\lambda) < 0 < \mu_+(\lambda)$ such that

$$g(\mu_\pm(\lambda);\lambda) = 0. \tag{1.9}$$

The main results of the current paper can then be stated as follows.

• (Tail properties of traveling wave solutions) *Suppose that* $v(\xi) = \Phi(\xi)$ *is a bounded positive solution of* (1.6). *Then*

$$\limsup_{\xi\to\pm\infty} \frac{\Phi(\xi)}{e^{\mu_\mp \xi}} < \infty,$$

where $\mu_\mp = \mu_\mp(0)$ (see Theorem 3.1).

• (Equivalence of persistence and existence of traveling wave solutions) *The following two statements are equivalent: persistence occurs in* (1.1) *and there are traveling wave solutions of* (1.1) *with speed c, which implies that the following two statements are equivalent: extinction occurs in* (1.1) *and there are no traveling wave solutions of* (1.1) *with speed c* (see Theorem 4.1 and Remark 4.1).

• (Existence, uniqueness, and nonexistence of traveling wave solutions) *There is* $0 \le L^* \le \infty$ *such that if* $L^* < L < \infty$, *then* (1.5) *has a unique positive stationary solution* $v(t,\xi) = \Phi(\xi)$. *On the other hand, if* $0 < L < L^*$, *then there is no positive stationary solution of* (1.5). *Moreover, if* $0 \le c < c^*$, *then* $0 \le L^* < \infty$, *and if* $c \ge c^*$, *then* $L^* = \infty$ (see Theorems 5.1 and 5.2).

Observe that persistence and extinction in (1.5) are strongly related to the spectral problem of the linearization of (1.5) at the trivial solution $v \equiv 0$,

$$\frac{\partial v(t,\xi)}{\partial t} = c\frac{\partial v(t,\xi)}{\partial \xi} + \int_{\mathbb{R}} \kappa(\eta - \xi) v(t,\eta) d\eta - v(t,\xi) + f(\xi,0)v(t,\xi), \quad \xi \in \mathbb{R}. \tag{1.10}$$

Let $\lambda(c,L)$ be the principal spectral point (see Definition 6.1) of the spectral problem associated with (1.10). We also prove that

• (Principal eigenvalue) $\lambda(c,L)$ *is a principal eigenvalue. It is continuous in* $(c,L) \in (0,\infty)\times(0,\infty)$, *and for fixed* $c > 0$, *it is strictly increasing in* $L > 0$ *(see Theorem 6.2(1)–(3)). Moreover, if* $0 < c < c^*$, *then there is* $0 \le L^{**} \le \infty$ *such that* $\lambda(c,L) > 0$ *for all* $L > L^{**}$, *and for any* $0 < L < L^{**}$, $\lambda(c,L) \le 0$. *If* $c > c^*$, *then* $\lambda(c,L) < 0$ *for all* $L > 0$ (see Theorem 6.2(4)–(5)).

• (Persistence/extinction) *If* $\lambda(c,L) > 0$, *then there is a positive stationary solution of* (1.5), *and for any* $K > 0$ *and* $u_0 \in X^{++} := \{u \in X | u > 0\}$ *satisfying* $\liminf_{\xi\to\infty} \frac{u_0(\xi)}{e^{\mu_-(\lambda(c,L))\xi}} > 0$ *and* $\liminf_{\xi\to-\infty} \frac{u_0(\xi)}{e^{\mu_+(\lambda(c,L))\xi}} > 0$,

$$\liminf_{t\to\infty} \inf_{|\xi|\le K} v(t,\xi;u_0) > 0,$$

where $v(t,\xi;u_0)$ *is the solution of* (1.5) *with* $v(0,\xi;u_0) = u_0(\xi)$, *and* $\mu_\pm(\lambda(c,L))$ *are as in* (1.9). *If* $\lambda(c,L) \le 0$, *then for any* $u_0 \in X^+$,

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} v(t,\xi;u_0) = 0.$$

(see Theorem 6.3).

• (Dependence of L^* and L^{**} on c) $L^* = L^{**}$, and $L^* \to \infty$ as $c \to (c^*)^-$ (see Corollary 6.1).

We conclude the introduction with the following three remarks.

First, as was pointed above, the specific form of $f(x,u)$ in the transition region in (H2) can be replaced by a more general form. But, in general, the number L^* depends on the form which f takes in the transition region.

Second, for fixed $L > 0$, it remains an open problem whether $\lambda(c,L)$ is monotone in $c > 0$.

Third, in the case $\lambda(c,L) > 0$, due to the lack of compactness of $v(t,\xi;u_0)$ for $u_0 \in X^{++}$, it is a nontrivial problem to study the convergence of $v(t,\xi;u_0)$ as $t \to \infty$. The reader is referred to [8] and [9] for such a convergence study in random dispersal equations, and integrodifference equations with a shifting climate, respectively.

The rest of the paper is organized as follows. In Section 2, we present some preliminaries such as comparison principles for nonlocal evolution equations. In Section 3, we show the tail behaviors of the traveling waves. We examine in Section 4 the equivalence of the occurrence of persistence and the existence of traveling wave solutions. In Section 5, we prove the existence, uniqueness and nonexistence of the traveling wave solutions. In Section 6, we investigate the spectral theory of nonlocal operators and discuss their applications to species persistence and extinction.

2. Preliminaries

In this section, we present some preliminary materials to be used in the following sections.

2.1. Comparison principle of nonlocal evolution equations

A continuous function $v(t,\xi)$ on $[0,T)\times\mathbb{R}$ is called a *super-solution* or *sub-solution* of (1.5) if $\frac{\partial v}{\partial t}, \frac{\partial v}{\partial \xi}$ exist and are continuous on $[0,T)\times\mathbb{R}$ and satisfy

$$\frac{\partial v}{\partial t} \geq c\frac{\partial v}{\partial \xi} + \int_{\mathbb{R}} k(\eta-\xi)v(t,\eta)d\eta - v(t,\xi) + f(\xi,v)v(t,\xi), \quad \xi \in \mathbb{R}$$

or

$$\frac{\partial v}{\partial t} \leq c\frac{\partial v}{\partial \xi} + \int_{\mathbb{R}} k(\eta-\xi)v(t,\eta)d\eta - v(t,\xi) + f(\xi,v)v(t,\xi), \quad \xi \in \mathbb{R}$$

for $t \in [0,T)$, respectively. The super-/sub-solutions for the linear equation (1.10) are defined similarly.

Proposition 2.1 (*Comparison Principle*)**.**

(1) If $\underline{v}(t,\xi)$ and $\overline{v}(t,\xi)$ are sub-solution and super-solution of (1.10) on $[0,T)$, respectively, $\underline{v}(0,\cdot) \leq \overline{v}(0,\cdot)$, and $\overline{v}(t,\xi)-\underline{v}(t,\xi) \geq -\beta_0$ for $(t,\xi)\in[0,T)\times\mathbb{R}$ and some $\beta_0 > 0$, then $\underline{v}(t,\cdot) \leq \overline{v}(t,\cdot)$ for $t \in [0,T)$.

(2) Suppose that $v_1, v_2 \in X$ and $v_1 \leq v_2$, $v_1 \neq v_2$. Then $v(t,\xi;v_1) < v(t,\xi;v_2)$ for all $t > 0, \xi \in \mathbb{R}$, where $v(t,\xi;v_k)$ is the solution of (1.5) with $v(0,\xi;v_k) = v_k$ for $k = 1,2$.

Proof. (1) This follows by modifying the arguments in [4, Proposition 2.1]. Let $v(t,\xi) = e^{\sigma t}(\overline{v}(t,\xi)-\underline{v}(t,\xi))$. Then $v(t,\xi) \geq -e^{\sigma t}\beta_0$, and

$$\frac{\partial v}{\partial t} \geq cv_\xi(t,\xi) + \int_{\mathbb{R}} k(\eta-\xi)v(t,\eta)d\eta + p(\xi)v(t,\xi), \quad \xi \in \mathbb{R}, \tag{2.1}$$

for $t \in (0,T)$ and $p(\xi) = f(\xi,0)-1+\sigma$. Choose $\sigma > 0$ such that $p(\xi) > 0$ for all $(t,\xi)\in[0,T)\times\mathbb{R}$. We claim that $v(t,\xi) \geq 0$ for $(t,\xi)\in[0,T)\times\mathbb{R}$.

Let $p_0 = \sup\limits_{\xi\in\mathbb{R}} p(\xi)$. Let $T_0 = \min\{T, \frac{1}{p_0+1}\}$. Let $\xi = \hat{x}-ct$ and $\eta = \hat{y}-ct$, then

$$\frac{\partial v(t,\hat{x}-ct)}{\partial t} \geq \int_{\mathbb{R}} k(\hat{y}-\hat{x})v(t,\hat{y}-ct)d\hat{y} + p(\hat{x}-ct)v(t,\hat{x}-ct), \quad \hat{x} \in \mathbb{R}, \tag{2.2}$$

for $t \in (0,T)$.

Assume that there are $\tilde{t} \in (0, T_0)$ and $\tilde{x} \in \mathbb{R}$ such that $v(\tilde{t}, \tilde{x}) < 0$. Let

$$v_{\inf} := \inf_{(t,\hat{x})\in[0,\tilde{t}]\times\mathbb{R}} v(t, \hat{x} - ct) < 0.$$

Observe that there are $t_n \in (0, \tilde{t}]$ and $x_n \in \mathbb{R}$ such that

$$v(t_n, x_n - ct_n) \to v_{\inf} \quad \text{as} \quad n \to \infty.$$

By (2.2), we have that

$$\begin{aligned} v(t_n, x_n - ct_n) - v(0, x_n) &\geq \int_0^{t_n} \Big[\int_{\mathbb{R}} k(\hat{y} - x_n)v(t, \hat{y} - ct)d\hat{y} + p(x_n - ct)v(t, x_n - ct)\Big]dt \\ &\geq \int_0^{t_n} \Big[\int_{\mathbb{R}} k(\hat{y} - x_n)v_{\inf}d\hat{y} + p_0 v_{\inf}\Big]dt \\ &= t_n(1 + p_0)v_{\inf} \\ &\geq \tilde{t}(1 + p_0)v_{\inf} \end{aligned}$$

for $n = 1, 2, \ldots$. Note that $v(0, x_n) \geq 0$ for $n = 1, 2, \ldots$. We then have that

$$v(t_n, x_n - ct_n) \geq \tilde{t}(1 + p_0)v_{\inf}$$

for $n = 1, 2, \ldots$. Letting $n \to \infty$, since $\tilde{t}(1 + p_0) < T_0(1 + p_0) \leq \frac{1}{1+p_0}(1 + p_0) = 1$ and $v_{\inf} < 0$, we get

$$v_{\inf} \geq \tilde{t}(1 + p_0)v_{\inf} > v_{\inf}.$$

This is a contradiction, that implies that $v(t, \xi) \geq 0$ for $(t, \xi) \in [0, T_0) \times \mathbb{R}$. The above procedure can be repeated for $t \in [kT_0, (k + 1)T_0) \bigcap [0, T)$ for $k = 1, 2....$ Hence $v(t, \xi) \geq 0$ for $(t, \xi) \in [0, T) \times \mathbb{R}$ and then $\underline{u}(t, \xi) \leq \overline{v}(t, \xi)$ for $(t, \xi) \in [0, T) \times \mathbb{R}$.

(2) This follows from similar arguments as in [4, Proposition 2.2]. □

2.2. Comparison principle for nonlocal Dirichlet boundary problems

In this subsection, we consider the following linear nonlocal equations with non-homogeneous Dirichlet boundary conditions:

$$\begin{cases} \frac{\partial\varphi(t,\xi)}{\partial t} = c\frac{\partial\varphi(t,\xi)}{\partial\xi} + \int_{\mathbb{R}} k(\eta - \xi)\varphi(t, \eta)d\eta - \varphi(t, \xi) + q(\xi)\varphi(t, \xi), \ \xi \in (a, b) \\ \varphi(t, \xi) = g(\xi), \ \xi \notin (a, b) \end{cases} \tag{2.3}$$

for $a, b \in \mathbb{R}$, $b > a$, $q(\xi)$, $g(\xi) \in X$.

Let $\mathcal{L}_D v := cv'(\xi) + \int_{\mathbb{R}} k(\eta - \xi)v(\eta)d\eta - v(\xi) + q(\xi)v(\xi)$ for $v, v' \in X$. A function $v(t, \xi)$ is called a *super-solution* or *sub-solution* of (2.3) for $t \in [t_0, t_0 + T]$ if $v(t, \cdot), v_\xi(t, \cdot) \in X$ and v satisfies

$$\begin{cases} v_t(t, \xi) \geq \mathcal{L}_D v, \quad \xi \in (a, b) \\ v(t, \xi) \geq g(\xi), \ \xi \notin (a, b) \end{cases}$$

or

$$\begin{cases} v_t(t, \xi) \leq \mathcal{L}_D v, \quad \xi \in (a, b) \\ v(t, \xi) \leq g(\xi), \ \xi \notin (a, b), \end{cases}$$

respectively, for $t \in [t_0, t_0 + T]$.

Proposition 2.2. *Let $\overline{u}$ and $\underline{v}$ be the super-solution and sub-solution of (2.3) with boundary conditions $\overline{g}, \underline{g}$ respectively. Suppose that $\bar{u}(t_0,\xi) \geq \underline{v}(t_0,\xi)$ for $\xi \in [a,b]$, $\overline{g}(\xi) \geq \underline{g}(\xi)$ for $\xi \notin (a,b)$ and $q(\xi) < 0$ for $\xi \in (a,b)$. Then $\overline{u}(t,\xi) \geq \underline{v}(t,\xi)$ for $t \in [t_0, t_0+T]$ and $\xi \in [a,b]$.*

Proof. Let $w = \overline{u} - \underline{v}$. Then

$$\mathcal{L}_D w = \mathcal{L}_D \overline{u} - \mathcal{L}_D \underline{v} \leq w_t.$$

We claim that $w \geq 0$. Suppose not, then there are $t^0 \in [t_0, t_0+T]$, $\xi_0 \in [a,b]$ such that $w(t^0,\xi_0) = \min_{t\in[t_0,t_0+T],\xi\in[a,b]} \{w(t,\xi)\} < 0$. By $\overline{g}(\xi) \geq \underline{g}(\xi)$ for $\xi \notin (a,b)$, $w(t_0,\xi) \geq 0$ for $\xi \in [a,b]$, we have that $t^0 \in (t_0, t_0+T]$ and $\xi_0 \in (a,b)$, and then $w_t(t^0,\xi_0) \leq 0$ and $w_\xi(t^0,\xi_0) = 0$. Therefore, since $w(t^0,\eta) - w(t^0,\xi_0) \geq 0$ for $\eta \in (a,b)$, $w(t^0,\xi_0) < 0$ and $q(\xi_0) < 0$, there holds

$$\int_a^b k(\xi_0-\eta)[w(t^0,\eta) - w(t^0,\xi_0)]d\eta + \int_{\mathbb{R}\setminus(a,b)} k(\xi_0-\eta)d\eta w(t^0,\xi_0) + q(\xi_0)w(t^0,\xi_0) > 0.$$

Thus we have that $\mathcal{L}_D w(t^0,\xi_0) > 0$, which is a contradiction. □

2.3. Convergence on compact sets

In this subsection, we explore the convergence property of solutions of (1.10) and (1.5) in compact open topology. Note that $f(\xi,v)$ depends on r, q, L, and L_0. View L as a parameter and write $f(\xi,v)$ as $f(\xi,v;L)$. Let $f(\xi,v;\infty) = r(1-v)$. For fixed r, q, and L_0, to indicate the dependence of solutions of (1.10) and (1.5) on L, we denote the solution of (1.10) (or (1.5)) by $\tilde{v}(t,\xi;u_0,L)$ (or $v(t,\xi;u_0,L)$).

Proposition 2.3 (*Convergence on Compact Subsets*)**.** *Suppose that $u_{0n}, u_0 \in X^+$ ($n = 1,2,\ldots$) and $\{\|u_{0n}\|\}$ is bounded.*

(1) If as $n \to \infty$, $u_{0n}(\xi) \to u_0(\xi)$ uniformly in ξ on bounded sets and $L_n \to \infty$, then for each $t > 0$, $\tilde{v}(t,\xi;u_{0n},L_n) \to \tilde{v}_\infty(t,\xi;u_0)$ as $n \to \infty$ uniformly in ξ on bounded sets, where $\tilde{v}(t,\xi;u_{0n},L_n)$ is the solution of (1.10) with $\tilde{v}(0,\xi;u_{0n},L_n) = u_{0n}$ and $\tilde{v}_\infty(t,\xi;u_0)$ is the solution of

$$v_t = cv_\xi + \int_{\mathbb{R}} \kappa(\eta-\xi)v(t,\eta)d\eta - v(t,\xi) + rv(t,\xi), \quad \xi \in \mathbb{R} \tag{2.4}$$

with $\tilde{v}_\infty(0,\xi;u_0) = u_0(\xi)$ for $\xi \in \mathbb{R}$.

(2) Fix L. If $u_{0n}(\xi) \to u_0(\xi)$ as $n \to \infty$ uniformly in ξ on bounded sets, then for each $t > 0$, $v(t,\xi;u_{0n}) \to v(t,\xi;u_0)$ as $n \to \infty$ uniformly in ξ on bounded sets, where $v(t,\xi;u_{0n})$ and $v(t,\xi;u_0)$ are the solutions of (1.5) with $v(0,\xi;u_{0n}) = u_{0n}$ and $v(0,\xi;u_0) = u_0$ respectively.

Proof. It can be proved by the similar arguments as those in [17, Proposition 3.3]. For completeness, we provide a proof in the following.

(1) Let $v^n(t,\xi) = \tilde{v}(t,\xi;u_{0n},L_n) - \tilde{v}_\infty(t,\xi;u_0)$. Then $v^n(t,\xi)$ satisfies

$$v_t^n(t,\xi) = cv_\xi^n(t,\xi) + \int_{\mathbb{R}} \kappa(\eta-\xi)v^n(t,\eta)d\eta - v^n(t,\xi) + a_n(\xi)v^n(t,\xi) + b_n(t,\xi),$$

where $a_n(\xi) = f(\xi,0;L_n)$ and $b_n(t,\xi) = \tilde{v}_\infty(t,\xi;u_0)\big(a_n(\xi) - r\big)$.

Note that $\{a_n(\xi)\}$ is uniformly bounded and continuous in ξ with $|a_n(\xi)| \leq \max\{r,q\}$. By (H2), $b_n(t,\xi) \to 0$ as $n \to \infty$ uniformly in (t,ξ) on bounded sets of $[0,\infty) \times \mathbb{R}$.

Take a $\rho > 0$. Let

$$X(\rho) = \{u \in C(\mathbb{R},\mathbb{R}) \mid u(\cdot)e^{-\rho\|\cdot\|} \in X\}$$

with norm $\|u\|_\rho = \|u(\cdot)e^{-\rho\|\cdot\|}\|$. Let

$$\mathcal{A}u = cu' + \int_{\mathbb{R}} k(\eta-\xi)u(\eta)d\eta - u(\xi)$$

for $u \in X(\rho)$ with $u'(\cdot) \in X(\rho)$. Let $\mathcal{D} : X^1(\rho) \to X(\rho)$ be defined by $\mathcal{D}u = u'$ for $u \in X^1(\rho) := \{u \in X(\rho) | u'(\cdot) \in X(\rho)\}$. Since the resolvent, denoted by $R(\lambda, \mathcal{D})$, can be computed explicitly as

$$R(\lambda, \mathcal{D})u = \int_{-\infty}^{\xi} e^{-\lambda(\xi-s)}u(s)ds, \xi \in \mathbb{R}.$$

Moreover, $\|R(\lambda, \mathcal{D})\|_{X(\rho)} \le \frac{1}{\lambda+\rho}$ for all $\lambda > 0$. Thus, $\mathcal{D}$ is dissipative and $\lambda I - \mathcal{D}$ is surjective for $\lambda > 0$,where I denotes the identity operator. Hence, by Lumer–Phillips theorem (See Section 1.4 of [16]) $\mathcal{D}$ generates a C_0-semigroup on $X(\rho)$. Therefore, by perturbations of bounded linear operators (Theorem 1.1 in Section 3.1 of [16]), $\mathcal{A}$ generates a C_0-semigroup on $X(\rho)$ denoted by $e^{\mathcal{A}t}$, and there are $M > 0$ and $\omega > 0$ such that

$$\|e^{\mathcal{A}t}\|_{X(\rho)} \le Me^{\omega t} \quad \forall t \ge 0.$$

Hence

$$v^n(t,\cdot) = e^{\mathcal{A}t}v^n(0,\cdot) + \int_0^t e^{\mathcal{A}(t-\tau)}a_n(\cdot)v^n(\tau,\cdot)d\tau + \int_0^t e^{\mathcal{A}(t-\tau)}b_n(\tau,\cdot)d\tau$$

and then

$$\begin{aligned}\|v^n(t,\cdot)\|_{X(\rho)} \le\ & Me^{\omega t}\|v^n(0,\cdot)\|_{X(\rho)} + M\sup_{\xi\in\mathbb{R}}|a_n(\xi)| \int_0^t e^{\omega(t-\tau)}\|v^n(\tau,\cdot)\|_{X(\rho)}d\tau \\ & + M\int_0^t e^{\omega(t-\tau)}\|b_n(\tau,\cdot)\|_{X(\rho)}d\tau \\ \le\ & Me^{\omega t}\|v^n(0,\cdot)\|_{X(\rho)} + M\sup_{\xi\in\mathbb{R}}|a_n(\xi)| \int_0^t e^{\omega(t-\tau)}\|v^n(\tau,\cdot)\|_{X(\rho)}d\tau \\ & + \frac{M}{\omega}\sup_{\tau\in[0,t]}\|b_n(\tau,\cdot)\|_{X(\rho)}e^{\omega t}.\end{aligned}$$

By Gronwall's inequality,

$$\|v^n(t,\cdot)\|_{X(\rho)} \le exp\big((\omega + M\sup_{\xi\in\mathbb{R}}|a_n(\xi)|)t\big)\Big(M\|v^n(0,\cdot)\|_{X(\rho)} + \frac{M}{\omega}\sup_{\tau\in[0,t]}\|b_n(\tau,\cdot)\|_{X(\rho)}\Big).$$

Note that $\|v^n(0,\cdot)\|_{X(\rho)} \to 0$ and $\sup_{\tau\in[0,t]}\|b_n(\tau,\cdot)\|_{X(\rho)} \to 0$ as $n \to \infty$. It then follows that

$$\|v^n(t,\cdot)\|_{X(\rho)} \to 0 \quad \text{as} \quad n \to \infty$$

and then

$$\tilde{v}(t,\xi;u_{0n},L_n) \to \tilde{v}_\infty(t,\xi;u_0) \quad \text{as} \quad n \to \infty$$

uniformly in ξ on bounded sets.

(2) Let $v^n(t,\xi) = v(t,\xi;u_{0n}) - v(t,\xi;u_0)$. Then $v^n(t,\xi)$ satisfies

$$v^n_t(t,\xi) = cv^n_\xi(t,\xi) + \int_{\mathbb{R}} \kappa(\eta-\xi)v^n(t,\eta)d\eta - v^n(t,\xi) + a_n(t,\xi)v^n(t,\xi) + b_n(t,\xi),$$

where

$$a_n(t,\xi) = f(\xi, v(t,\xi;u_{0n})) + v(t,\xi;u_0)\cdot\int_0^1 \partial_u f(\xi, sv(t,\xi;u_{0n}) + (1-s)v(t,\xi;u_0))ds$$

and

$$b_n(t,\xi) = v(t,\xi;u_0)\cdot\big(f(\xi, v_\infty(t,\xi;u_0)) - f(\xi, v(t,\xi;u_0))\big).$$

By the boundedness of $\{\|u_{0n}\|\}$, $\{a_n(t,\xi)\}$ is uniformly bounded and continuous in t and ξ. By (H2), $b_n(t,\xi) \to 0$ as $n \to \infty$ uniformly in (t,ξ) on bounded sets of $[0,\infty)\times\mathbb{R}$. The rest follows from a simple modification of the proof in (1). $\square$

2.4. Spreading speeds in fixed habitats

In this subsection, we review some properties about the spreading speeds of the nonlocal dispersal equation (1.7). Let $\mu^* > 0$ and c^* satisfy that

$$c^* = \inf_{\mu>0} \frac{\int_{\mathbb{R}} e^{-\mu z} k(z)dz - 1 + r}{\mu} = \frac{\int_{\mathbb{R}} e^{-\mu^* z} k(z)dz - 1 + r}{\mu^*}. \tag{2.5}$$

Proposition 2.4.

(1) For any $u_0 \in X^+$ with nonempty compact support,

$$\limsup_{t\to\infty} \sup_{|x|\ge ct} u(t,x;u_0) = 0 \quad \forall\, c > c^*$$

and

$$\liminf_{t\to\infty} \inf_{|x|\le ct} (u(t,x;u_0) - 1) = 0 \quad \forall\, 0 < c < c^*,$$

where $u(t,x;u_0)$ is the solution of (1.7) with $u(0,x;u_0) = u_0(x)$.

(2) For any $c \ge c^*$, (1.7) has a positive traveling wave solution $u(t,x) = \phi(x-ct)$ with $\phi(-\infty) = 1$ and $\phi(\infty) = 0$. Moreover, for $\mu \in (0,\mu^*)$ such that $c = \frac{\int_{\mathbb{R}} e^{-\mu z} k(z)dz - 1 + r}{\mu}$, $\lim_{x\to\infty} \frac{\phi(x)}{e^{-\mu x}} = 1$.

Proof. (1) See Theorem E of [4].

(2) See Theorem 2.4 of [5]. □

3. Tail behavior of traveling waves

In this section, we study the tail behavior or decay behavior of positive stationary solutions of (1.6) (assuming they exist), or equivalently, traveling wave solutions of (1.1). The main result of this section can then be stated as follows.

Theorem 3.1. *Suppose that Φ is a bounded positive solution of (1.6). Then there are $M^\pm$ such that*

$$\limsup_{\xi\to\infty} \frac{\Phi(\xi)}{e^{\mu_- \xi}} \le M^+$$

and

$$\limsup_{\xi\to-\infty} \frac{\Phi(\xi)}{e^{\mu_+ \xi}} \le M^-,$$

where $\mu_\pm = \mu_\pm(0)$ and $\mu_\pm(\lambda)$ is as in (1.9).

To prove the above theorem, we first prove a lemma.

For given $M > 0$, $R^+ > L + L_0$, $R^- < -(L + L_0)$, and $\tau > 0$, consider the following nonlocal Dirichlet problems,

$$\begin{cases} cv'(\xi) + \int_{\mathbb{R}} k(\xi-\eta)v(\eta)d\eta - v(\xi) - qv(\xi) = 0, & R^+ < \xi < R^+ + \tau \\ v(\xi) = M, \ \xi \notin (R^+, R^+ + \tau), \end{cases} \tag{3.1}$$

and

$$\begin{cases} cv'(\xi) + \int_{\mathbb{R}} k(\xi-\eta)v(\eta)d\eta - v(\xi) - qv(\xi) = 0, & R^- - \tau < \xi < R^- \\ v(\xi) = M, \ \xi \notin (R^- - \tau, R^-). \end{cases} \tag{3.2}$$

Let

$$\psi_\tau^+(\xi) = k_1^+ e^{\mu_-(\xi - R^+)} + k_2^+ e^{\mu_+(\xi - R^+)}, \tag{3.3}$$

and

$$\psi_\tau^-(\xi) = k_1^- e^{\mu_-(\xi - R^-)} + k_2^- e^{\mu_+(\xi - R^-)}, \tag{3.4}$$

where $k_1^+ = M\frac{e^{\mu_+\tau}-1}{e^{\mu_+\tau}-e^{\mu_-\tau}}$, $k_2^+ = M\frac{1-e^{\mu_-\tau}}{e^{\mu_+\tau}-e^{\mu_-\tau}}$, and $k_1^- = M\frac{1-e^{-\mu_+\tau}}{e^{-\mu_-\tau}-e^{-\mu_+\tau}}$, $k_2^- = M\frac{e^{-\mu_-\tau}-1}{e^{-\mu_-\tau}-e^{-\mu_+\tau}}$.

Lemma 3.1. *$\psi_\tau^+(\xi)$ is a super-solution of (3.1) and $\psi_\tau^-(\xi)$ is a super-solution of (3.2), that is, they are super-solutions of (2.3) with $a = R^+$, $b = R^+ + \tau$, and $g(\xi) \equiv M$, and $a = R^- - \tau$, $b = R^-$, and $g(\xi) \equiv M$, respectively.*

Proof. Consider

$$cv'(\xi) + \int_{\mathbb{R}} k(\xi - \eta)v(\eta)d\eta - v(\xi) - qv(\xi) = 0. \tag{3.5}$$

Let $v(\xi) = e^{\mu\xi}$ and then the characteristic equation of (3.5) becomes

$$c\mu + \int_{\mathbb{R}} e^{\mu(\eta-\xi)} k(\xi - \eta)d\eta - 1 - q = g(\mu; 0) = 0, \tag{3.6}$$

where

$$g(\mu; 0) = c\mu + \int_{\mathbb{R}} e^{\mu\eta} k(\eta)d\eta - 1 - q.$$

Note that $g(\mu_\pm; 0) = 0$. Thus, $\psi(\xi) = k_1 e^{\mu_-\xi} + k_2 e^{\mu_+\xi}$ is a solution of (3.5) for any choice of the scalars k_1 and k_2.

Let ψ_τ^+ be as in (3.3). Note that $\psi_\tau^+(R^+) = \psi_\tau^+(R^+ + \tau) = M$. Then there exists a $\theta \in (R^+, R^+ + \tau)$ such that $(\psi_\tau^+)'(\theta) = 0$. $(\psi_\tau^+)''(\xi) > 0$ implies that $(\psi_\tau^+)'(\xi) < 0$ for $\xi < \theta$ and $(\psi_\tau^+)'(\xi) > 0$ for $\xi > \theta$. Therefore $\psi_\tau^+(\xi) \geq M$ for $\xi \leq R^+$ and $\xi \geq R^+ + \tau$. By definition, $\psi_\tau^+(\xi)$ is a super-solution of (3.1).

Similarly, we can prove that $\psi_\tau^-(\xi)$ is a super-solution of (3.2). □

We now prove Theorem 3.1.

Proof of Theorem 3.1. Choose $M = \max_\xi \Phi(\xi)$. Observe that $v(t, \xi) = \Phi(\xi)$ is a sub-solution of (2.3) with $a = R^+$, $b = R^+ + \tau$, and $g(\xi) \equiv M$ for any $R^+ > L + L_0$ and any $\tau > 0$. Then by Lemma 3.1 and Proposition 2.2, for any given $R^+ > L + L_0$ and $\tau > 0$, $\Phi(\xi) \leq \psi_\tau^+(\xi)$ for $\xi \in (R^+, R^+ + \tau)$. Note that

$$\lim_{\tau\to\infty} \psi_\tau^+(\xi) = Me^{\mu_-(\xi - R^+)}.$$

We then have that

$$\Phi(\xi) \leq Me^{\mu_-(\xi - R^+)} \quad \forall\, \xi > R^+, \tag{3.7}$$

and thus

$$0 \leq \liminf_{\xi\to\infty} \frac{\Phi(\xi)}{e^{\mu_-\xi}} \leq \limsup_{\xi\to\infty} \frac{\Phi(\xi)}{e^{\mu_-\xi}} \leq M^+ := Me^{-\mu_- R^+}.$$

Similarly, we have that

$$\Phi(\xi) \leq Me^{\mu_+(\xi - R^-)} \quad \forall\, \xi < R^-, \tag{3.8}$$

and thus

$$0 \leq \liminf_{\xi\to-\infty} \frac{\Phi(\xi)}{e^{\mu_+\xi}} \leq \limsup_{\xi\to-\infty} \frac{\Phi(\xi)}{e^{\mu_+\xi}} \leq M^- := Me^{-\mu_+ R^-}.$$

Remark 3.1. In general, it remains an open question whether the limits $\lim_{\xi\to\infty} \frac{\Phi(\xi)}{e^{\mu_-\xi}}$ and $\lim_{\xi\to-\infty} \frac{\Phi(\xi)}{e^{\mu_+\xi}}$ exist.

4. Equivalence of the persistence and the existence of traveling wave solutions

In this section, we show that the occurrence of persistence and the existence of traveling wave solutions are equivalent.

Theorem 4.1. *The following two statements are equivalent:*

(1) Persistence occurs in (1.1)
(2) There are traveling wave solutions of (1.1) with speed c.

Proof. First, we prove that (2) implies (1).

Suppose that (2) holds and that Φ is a bounded positive solution of (1.6). Then $\tilde{v}(t,\xi;\gamma\Phi) := \gamma\Phi(\xi)$ satisfies

$$v_t = cv_\xi + \int_{\mathbb{R}} k(\eta-\xi)v(t,\eta)d\eta - v(t,\xi) + f(\xi,\Phi(\xi))v(t,\xi), \quad \xi\in\mathbb{R} \tag{4.1}$$

for any $\gamma\in\mathbb{R}$. Let $v(t,\xi;\gamma\Phi)$ be the solution of (1.5) with $v(0,\xi;\gamma\Phi)=\gamma\Phi(\xi)$. Fix $T>0$. Note that $\inf_{|\xi|\le L+L_0}\Phi(\xi)>0$ and $v(t,\xi;0)\equiv 0$ for all $t\ge 0$ and $\xi\in\mathbb{R}$. Then there is $\gamma_0>0$ such that for any $0<\gamma<\gamma_0$,

$$v(t,\xi;\gamma\Phi)\le \Phi(\xi) \quad \text{for} \quad 0\le t\le T,\ |\xi|\le L+L_0.$$

This implies that for any $0<\gamma<\gamma_0$,

$$f(\xi,\Phi(\xi))\le f(\xi,v(t,\xi;\gamma\Phi)) \quad \text{for} \quad 0\le t\le T,\ \xi\in\mathbb{R}.$$

Thus $v(t,\xi)=v(t,\xi;\gamma\Phi)$ is a super-solution of (4.1) because

$$\begin{aligned}
&v_t(t,\xi) - [cv_\xi + \int_{\mathbb{R}} k(\eta-\xi)v(t,\eta)d\eta - v(t,\xi) + f(\xi,\Phi(\xi))v(t,\xi)]\\
&= v_t - [cv_\xi + \int_{\mathbb{R}} k(\eta-\xi)v(t,\eta)d\eta - v(t,\xi) + f(\xi,v)v(t,\xi)] + (f(\xi,v)-f(\xi,\Phi(\xi)))v(t,\xi)\\
&= (f(\xi,v)-f(\xi,\Phi(\xi)))v(t,\xi)\\
&\ge 0.
\end{aligned}$$

Note that $v(t,\xi;\gamma\Phi)-\tilde{v}(t,\xi;\gamma\Phi)\ge -\beta_0 := -\max_\xi\{\gamma\Phi(\xi)\}$ and then by the comparison principle (see Proposition 2.1(1)) applied for (4.1), we have that for any $0<\gamma<\gamma_0$,

$$v(t,\xi;\gamma\Phi)\ge \tilde{v}(t,\xi;\gamma\Phi)=\gamma\Phi(\xi), \quad \forall\, 0\le t\le T,\ \xi\in\mathbb{R}$$

and then by the strong comparison principle (see Proposition 2.1(2))

$$v(t,\xi;\gamma\Phi)>\gamma\Phi(\xi), \quad \forall\, t>0,\ \xi\in\mathbb{R}. \tag{4.2}$$

Note that for any given $u_0\in X^+$ with $\inf_{\xi\in\mathbb{R}} u_0(\xi)>0$, there is $0<\gamma<\gamma_0$ such that

$$u_0(\xi)\ge\gamma\Phi(\xi), \quad \forall\xi\in\mathbb{R}.$$

Then by Proposition 2.1(2), we have that

$$v(t,\xi;u_0)\ge v(t,\xi;\gamma\Phi)\ge\gamma\Phi(\xi), \quad \forall\, t\ge 0,\ \xi\in\mathbb{R}.$$

This implies that

$$\liminf_{t\to\infty}\inf_{|\xi|\le K} v(t,\xi;u_0)>0$$

for any $K>0$. Therefore, (1) holds.

Next, we prove that (1) implies (2).

Assume (1) holds, that is, persistence occurs in (1.1). Choose $M \gg 1$ such that $u_M(\xi) \equiv M$ is a super-solution of (1.5). Then by the definition of persistence in (1.1)

$$\liminf_{t\to\infty} \inf_{|\xi|\le K} v(t,\xi;u_M) > 0$$

for any $K > 0$. By Proposition 2.1,

$$v(t,\xi;u_M) \le u_M(\equiv M), \quad \forall\, t \ge 0,\ \xi \in \mathbb{R}, \tag{4.3}$$

and for any $t_2 > t_1 \ge 0$,

$$v(t_2,\xi;u_M) = v(t_1,\xi;v(t_2-t_1,\cdot;M)) \le v(t_1,\xi;u_M) \le M, \quad \forall\, \xi \in \mathbb{R}. \tag{4.4}$$

It then follows that there is $\Phi(\xi)$ such that

$$\lim_{t\to\infty} v(t,\xi;u_M) = \Phi(\xi), \quad \forall\, \xi \in \mathbb{R}$$

and

$$\inf_{|\xi|\le K} \Phi(\xi) > 0, \quad \forall\, K > 0.$$

In the following, we show that $\Phi(\xi)$ is C^1 and $v(t,\xi) = \Phi(\xi)$ is a positive stationary solution of (1.5). To this end, let $u(t,x;u_M)$ be the solution of (1.1) with $u(0,x;u_M) = u_M$. Note that with $\xi = x - ct$,

$$v(t,\xi;u_M) = u(t,x;u_M).$$

It then suffices to show that $u_t(t,x;u_M)$ and $u_{tt}(t,x;u_M)$ are uniformly bounded on $\mathbb{R}^+ \times \mathbb{R}$, and $u_x(t,x;u_M)$ exists and is uniformly bounded and continuous in $t \ge 0$ and $x \in \mathbb{R}$.

By (H2), the right hand side of (1.1) with $u(t,x)$ being replaced by $u(t,x;u_M)$ is uniformly continuous in $t \ge 0$ and $x \in \mathbb{R}$. Hence $u_t(t,x;u_M)$ is uniformly continuous in $t \ge 0$ and $x \in \mathbb{R}$. This implies that the right hand side of (1.1) with $u(t,x)$ being replaced by $u(t,x;u_M)$ is differentiable in t and its derivative with respect to t is uniformly continuous in $t \ge 0$ and $x \in \mathbb{R}$. It then follows that $u_{tt}(t,x;u_M)$ exists and is uniformly bounded on $\mathbb{R}^+ \times \mathbb{R}$.

To show that $u_x(t,x;u_M)$ exists and is uniformly bounded and continuous in $t \ge 0$ and $x \in \mathbb{R}$, let

$$w(t,x) = u_t(t,x;u_M) + u(t,x;u_M) - f(x-ct,u(t,x;u_M))u(t,x;u_M). \tag{4.5}$$

By (1.1), we have that

$$w(t,x) = \int_{\mathbb{R}} k(y-x)u(t,y;u_M)dy.$$

This together with (H1) implies that $w(t,x)$ is differentiable in x and $w_x(t,x)$ is uniformly bounded and uniformly continuous. For fixed x, let $\phi(t;x) = u(t,x;u_M)$. By (4.5), $\phi(t;x)$ is the solution of

$$\begin{cases} \frac{d\phi}{dt} = -\phi + f(x-ct,\phi)\phi + w(t,x) \\ \phi(0;x) = w(0,x). \end{cases} \tag{4.6}$$

View x as a parameter in the initial value problem (4.6). Note that $-\phi + f(x-ct,\phi)\phi + w(t,x)$ and $w(0,x)$ are continuously differentiable in $x \in \mathbb{R}$. Then by the smooth dependence of solutions of ODEs on the parameters, $\phi(t;x)$ is differentiable in x (hence $u_x(t,x;u_M)$ exists), and $\psi(t;x) = \phi_x(t;x)$ satisfies

$$\begin{cases} \frac{d\psi}{dt} = \big(-1 + f_u(x-ct,\phi)\phi(t;x) + f(x-ct,\phi)\big)\psi + w_x(t,x) + f_x(x-ct,\phi)\phi(t;x) \\ \psi(0;x) = 0. \end{cases} \tag{4.7}$$

Let $\mu(t;x) = -1 + f_u(x - ct, \phi)\phi(t;x) + f(x - ct, \phi)$. By the variation of constants formula, we have that

$$\phi_x(t,x) = \psi(t;x) = \int_0^t e^{\int_\tau^t \mu(s;x)ds}\big(w_x(\tau,x) + f_x(x - c\tau, \phi)\phi(\tau;x)\big)d\tau. \tag{4.8}$$

It is not difficult to see from (4.8) that $\psi(t;x)$ is uniformly bounded and continuous in $t \geq 0$ and $x \in \mathbb{R}$. We then have that $u_x(t,x;u_M)$ exists and is uniformly bounded and continuous in $t \geq 0$ and $x \in \mathbb{R}$.

Recall that

$$v(t,\xi;u_M) = u(t,x;u_M).$$

We then have that $v_t(t,\xi;u_M)$, $v_{tt}(t,\xi;u_M)$, and $v_\xi(t,\xi;u_M)$ exist and are uniformly bounded and continuous in $t \geq 0$ and $\xi \in \mathbb{R}$. This implies that $\Phi'(\xi)$ exists, and

$$\lim_{t\to\infty} v(t,\xi;u_M) = \Phi(\xi),$$

$$\lim_{t\to\infty} v_t(t,\xi;u_M) = 0,$$

and

$$\lim_{t\to\infty} v_\xi(t,\xi;u_M) = \Phi'(\xi)$$

locally uniformly in $\xi \in \mathbb{R}$. It then follows that $v(t,\xi) = \Phi(\xi)$ is a positive stationary solution of (1.5). This proves that (1) implies (2). □

Remark 4.1. By Theorem 4.1, the following two statements are equivalent:

(1) Extinction occurs in (1.1)
(2) There are no traveling wave solutions of (1.1) with speed c.

5. Existence, uniqueness, and nonexistence of traveling wave solutions

In this section, we study the existence, uniqueness, and nonexistence of positive traveling wave solutions of (1.1) with speed c, or equivalently, of positive stationary solutions of (1.5). In the next result we refer to the spreading speed c^* which was defined in formula (2.5) in Section 2.4.

Theorem 5.1 (*Existence/nonexistence of Traveling Wave Solutions*). *Fix $r, q, L_0 > 0$.*

(1) For given $0 \leq c < c^*$, there is $L^* \geq 0$ such that for $L > L^*$, (1.5) has a positive stationary solution $v(t,\xi) = \Phi(\xi)$. For $0 < L < L^*$, there is no positive stationary solution of (1.5).
(2) For given $c > c^*$, for any $L > 0$, there is no positive stationary solution of (1.5).

Proof. First of all, we note that $f(x,u)$ depends on r, q, L, and L_0. For clarity, for fixed r, q, and L_0, we write $f(x,u)$ as $f(x,u;L)$. By (H2), for $0 < L_1 < L_2$, we have that

$$f(x,u;L_1) \leq f(x,u,L_2), \quad \text{for} \quad 0 \leq u \leq 1,\ x \in \mathbb{R}. \tag{5.1}$$

Next, let $\bar{u}(x) \equiv 1$. We have that $v(t,x) \equiv 1$ is a super-solution of (1.5). Let $v(t,x;\bar{u},L)$ be the solution of (1.5) with $v(0,x;\bar{u},L) = \bar{u}(x) \equiv 1$. By the arguments in the proof of Theorem 4.1, there is $\Phi(x;L)$ such that

$$\lim_{t\to\infty} v(t,x;\bar{u},L) = \Phi(x;L) \leq 1$$

locally uniformly in $x \in \mathbb{R}$, and $v(x) = \Phi(x;L)$ is a nonnegative stationary solution of (1.5). Moreover, note that $u \equiv 0$ is a solution of (1.5). Then by Proposition 2.1, we have that either $\Phi(x;L) \equiv 0$, or $\Phi(x;L)$ is a positive stationary solution of (1.5). By (5.1) and Proposition 2.1, for any $0 < L_1 < L_2$, we have that

$$v(t,x;\bar{u},L_1) \leq v(t,x;\bar{u},L_2), \quad \forall\, t \geq 0,\ x \in \mathbb{R}.$$

Hence

$$\Phi(x;L_1)\le \Phi(x;L_2),\quad \forall\, x\in\mathbb{R},\ L_1\le L_2.$$

Therefore, there is $0\le L^*\le\infty$ such that

$$\begin{cases}\Phi(x;L)\equiv 0, & \forall\, 0<L\le L^*\\ \Phi(x;L)>0, & \forall\, L>L^*.\end{cases}$$

We claim that any nonnegative stationary solution $v^*(x)$ of (1.5) satisfies that

$$v^*(x)\le 1,\quad \forall\, x\in\mathbb{R}.$$

In fact, suppose that $v^*(x)$ is a nonnegative stationary solution of (1.5). Let $M=\sup_{x\in\mathbb{R}} v^*(x)$. If $M>1$, then $v(t,x)\equiv M$ is a super-solution of (1.5). By the comparison principle (Proposition 2.1),

$$v^*(x)\le v(t,x;M)<M,\quad \forall\, t>0,\ x\in\mathbb{R}.$$

This together with the tail property (Theorem 3.1) implies that

$$M=\sup_{x\in\mathbb{R}} v^*(x)=\max_{x\in\mathbb{R}} v^*(x)<M,$$

which is a contradiction. Hence $M\le 1$. It then follows that

$$v^*(x)\le v(t,x;\bar u,L),\quad \forall\, t>0,\ x\in\mathbb{R}$$

and then

$$v^*(x)\le \Phi(x;L).$$

Hence, if $\Phi(x;L)\equiv 0$, then (1.5) has no positive stationary solution.

We now prove (1) and (2).

(1) It suffices to prove that $L^*<\infty$. To this end, first, for $0<c<c^*$, choose $c'\in(c,c^*)$ and fix it. For given $u_0\in X^+$ with nonempty and compact support $\mathrm{supp}(u_0)$, by Proposition 2.4,

$$\liminf_{t\to\infty}\inf_{|x|\le c't}(u_\infty(t,x;u_0)-1)=0,$$

where $u_\infty(t,x;u_0)$ is the solution of (1.7) with $u_\infty(0,x;u_0)=u_0(x)$. Then we have that

$$\liminf_{t\to\infty}\inf_{-(c'+c)t\le x\le(c'-c)t}(u_\infty(t,x+ct;u_0)-1)=0.$$

Next, choose u_0 to be C^1. View x as a parameter and by the smooth dependence of solutions of ODEs on the parameters, then $u_\infty(t,x;u_0)$ is also C^1 in x and $v_\infty(t,x;u_0):=u_\infty(t,x+ct;u_0)$ is the solution of (2.4) with $v_\infty(0,x;u_0)=u_0(x)$ and satisfies

$$\liminf_{t\to\infty}\inf_{-(c'+c)t\le x\le(c'-c)t}(v_\infty(t,x;u_0)-1)=0.$$

Now choose u_0 such that $u_0\le 1/2$. Then there is $T>0$ such that

$$v_\infty(T,x;u_0)>1/2,\quad x\in\mathrm{supp}(u_0).$$

Let $v(t,x;u_0,L)$ be the solution of (1.5) with $v(0,x;u_0,L)=u_0(x)$. By Proposition 2.3,

$$\lim_{L\to\infty} v(t,x;u_0,L)=v_\infty(t,x;u_0)$$

locally uniformly in $(t,x)\in\mathbb{R}^+\times\mathbb{R}$. Then

$$v(T,x;u_0,L)\geq u_0(x),\quad \text{for}\quad L\gg 1.$$

Then by the comparison principle (Proposition 2.1), we have that for $L\gg 1$,

$$v(mT,x;u_0,L)\geq v((m-1)T,x;u_0,L)\geq u_0(x),\quad \forall\, x\in\mathbb{R},\ m=1,2,\ldots.$$

Since $v(mT,x;\bar{u},L)\geq v(mT,x;u_0,L)$ and $\Phi(x;L)=\lim_{m\to\infty}v(mT,x;\bar{u},L)$, we have that

$$\Phi(x;L)\geq u_0(x),\quad \forall\, x\in\mathbb{R}$$

for $L\gg 1$. Hence $u(t,x)=\Phi(x;L)$ is a positive stationary solution of (1.5) when $L\gg 1$, and then $L^*<\infty$.

(2) It suffices to prove that $L^*=\infty$. To this end, let $\mu^*>0$ be such that

$$c^*=\frac{\int_{\mathbb{R}}e^{\mu^*\eta}k(\eta)d\eta-1+r}{\mu^*}=\inf_{\mu>0}\frac{\int_{\mathbb{R}}e^{\mu\eta}k(\eta)d\eta-1+r}{\mu}.$$

Recall that $\mu_-<0<\mu_+$ are such that

$$c\mu_\pm+\int_{\mathbb{R}}e^{\mu_\pm\eta}k(\eta)d\eta-1-q=0. \tag{5.2}$$

We claim that $\mu^*<|\mu_-|$.

Indeed, let

$$h(\mu)=-c\mu+\int_{\mathbb{R}}e^{\mu\eta}k(\eta)d\eta-1-q.$$

Then $h(-\mu_-)=0$, $h'(\mu)>0$ for $\mu\geq-\mu_-$. Let

$$\tilde{h}(\mu)=\frac{\int_{\mathbb{R}}e^{\mu\eta}k(\eta)d\eta-1-q}{\mu}$$

for $\mu>0$. Note that $\int_{\mathbb{R}}\eta^m k(\eta)d\eta=0$ for odd integers $m=1,3,5,\ldots$ and we have that

$$\begin{aligned}
\tilde{h}'(\mu)&=\frac{\int_{\mathbb{R}}\eta e^{\mu\eta}k(\eta)d\eta\cdot\mu-(\int_{\mathbb{R}}e^{\mu\eta}k(\eta)d\eta-1-q)}{\mu^2}\\
&=\frac{1}{\mu^2}\Big[\int_{\mathbb{R}}(\eta^2k(\eta)\mu^2+\frac{\eta^4}{3!}k(\eta)\mu^4+\frac{\eta^6}{5!}k(\eta)\mu^6+\cdots)d\eta\\
&\quad-\int_{\mathbb{R}}(k(\eta)+\frac{\eta^2}{2!}k(\eta)\mu^2+\frac{\eta^4}{4!}k(\eta)\mu^4+\cdots)d\eta+1+q\Big]\\
&=\frac{1}{\mu^2}\Big[q+\int_{\mathbb{R}}((1-\frac{1}{2!})\eta^2k(\eta)\mu^2+(\frac{1}{3!}-\frac{1}{4!})\eta^4k(\eta)\mu^4+(\frac{1}{5!}-\frac{1}{6!})\eta^6k(\eta)\mu^6+\cdots)d\eta\Big]\\
&>0,\quad \text{for}\quad \mu>0.
\end{aligned}$$

Then $\mu=-\mu_-=|\mu_-|>0$ is the only solution of $\tilde{h}(\mu)=c$ in the interval $(0,\infty)$, and

$$\begin{aligned}
\tilde{h}(\mu^*)&=\frac{\int_{\mathbb{R}}e^{\mu^*\eta}k(\eta)d\eta-1-q}{\mu^*}\\
&<\frac{\int_{\mathbb{R}}e^{\mu^*\eta}k(\eta)d\eta-1+r}{\mu^*}=c^*\\
&<c=\frac{\int_{\mathbb{R}}e^{\mu_-\eta}k(\eta)d\eta-1-q}{-\mu_-}\quad \text{(by Eq. (5.2))}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\int_{\mathbb{R}} e^{-|\mu_-|\eta} k(\eta) d\eta - 1 - q}{|\mu_-|} \\
&= \frac{\int_{\mathbb{R}} e^{|\mu_-|\eta} k(\eta) d\eta - 1 - q}{|\mu_-|} \quad (\text{by } k(\eta) = k(-\eta)) \\
&= \tilde{h}(|\mu_-|).
\end{aligned}
$$

It then follows that $|\mu_-| > \mu^*$.

We now assume that (1.5) has a positive stationary solution $u = \Phi(\xi)$. By Theorem 3.1, there is $M^+ > 0$ such that

$$\Phi(\xi) \le M^+ e^{-|\mu_-|\xi}, \quad \text{for} \quad \xi \gg 1.$$

Choose $\tilde{c} \in (c^*, c)$. By Proposition 2.4, (1.7) has a traveling wave solution $u(t, x) = \phi(x - \tilde{c}t)$ such that

$$\phi(-\infty) = 1, \phi(+\infty) = 0, \text{and} \lim_{x \to \infty} \frac{\phi(x)}{e^{-\tilde{\mu}x}} = 1, \tag{5.3}$$

where $0 < \tilde{\mu} < \mu^* (< |\mu_-|)$ is such that

$$\tilde{c} = \frac{\int_{\mathbb{R}} e^{-\tilde{\mu}\eta} k(\eta) d\eta - 1 + r}{\tilde{\mu}}.$$

This implies that $v(t, x) = \phi(x - (\tilde{c} - c)t)$ is a super-solution of (1.5). Then by (H2), $v(t, x; \gamma) = \gamma\phi(x - (\tilde{c} - c)t)$ is a super-solution of (1.5) for any $\gamma \ge 1$. By Theorem 3.1, there is $\gamma \ge 1$ such that

$$\Phi(x) \le \gamma\phi(x).$$

Hence, by the comparison principle (Proposition 2.1), we have that

$$\Phi(x) \le \gamma\phi(x - (\tilde{c} - c)t), \quad \forall\, x \in \mathbb{R},\ t \ge 0.$$

Letting $t \to \infty$, since $\tilde{c} < c$ we have that $x - (\tilde{c} - c)t \to \infty$ and so by (5.3), $\lim_{t\to\infty} \phi(x - (\tilde{c} - c)t) = \phi(\infty) = 0$, implying that

$$\Phi(x) \le 0,\ \forall\, x \in \mathbb{R},$$

which is a contradiction.

Hence for any $L > 0$, (1.5) has no positive stationary solution and then $L^* = \infty$. □

The following corollary follows directly from the proof of the above theorem.

Corollary 5.1. *Suppose that there is no positive traveling wave solution of (1.1). Then for any $u_0 \ge 0$,*

$$\lim_{t\to\infty} v(t, x; u_0) = 0$$

locally uniformly in $x \in \mathbb{R}$, where $v(t, x; u_0)$ is the solution of (1.5) with $v(0, x; u_0) = u_0(x)$.

We will prove the following theorem about the uniqueness of traveling waves of (1.1) by modifying the proof of Theorem 2.1 in [18], where the authors dealt with the uniqueness of forced waves (traveling waves) for nonlocal equation (1.1) with nonlinearity (1.4). Due to the different nonlinearity and tail properties of our traveling waves, their proof cannot be applied directly.

Theorem 5.2 (*Uniqueness*). *There are at most one positive bounded solution to Eq. (1.6).*

Proof. Let $\Phi_i(\xi), i = 1, 2$ be two positive bounded solutions of Eq. (1.1), that is, Φ_i satisfy that

$$c\Phi_i'(\xi) + \int_{\mathbb{R}} \kappa(\eta - \xi)\Phi_i(\eta)d\eta - \Phi_i(\xi) + f(\xi, \Phi_i)\Phi_i(\xi) = 0, \quad \xi \in \mathbb{R}, i = 1, 2.$$

Define $\Sigma_\epsilon = \{\sigma \geq 1 | \sigma\Phi_2 \geq \Phi_1 - \epsilon\}$ for $\epsilon > 0$. Note that $\Phi_1(\pm\infty) = 0$ and then $\Phi_1 - \epsilon$ is nonnegative only on a bounded region. Therefore there exists a large enough σ such that $\sigma\Phi_2 \geq \Phi_1 - \epsilon$, that is, Σ_ϵ is not empty. Let $\sigma_\epsilon = \inf \Sigma_\epsilon \geq 1$. Note that σ_ϵ is non-increasing in ϵ. Hence $\lim\limits_{\epsilon\to 0} \sigma_\epsilon$ exists. Let $\sigma^* = \lim\limits_{\epsilon\to 0} \sigma_\epsilon \in [1, \infty]$. We claim that $\sigma^* = 1$. If $\sigma^* = 1$, then we have that $\Phi_2(\xi) \geq \Phi_1(\xi)$. Repeat the previous process by interchanging Φ_1 and Φ_2, and then we also have that $\Phi_1(\xi) \geq \Phi_2(\xi)$. Thus $\Phi_1 \equiv \Phi_2$.

Now it suffices to prove the claim that $\sigma^* = 1$. Suppose to the contrary that $\sigma^* > 1$. Then $\sigma_{\epsilon_0} > 1$ for some ϵ_0. This implies that $\sigma_\epsilon > 1$ for all $0 < \epsilon \leq \epsilon_0$. Let $w_\epsilon(\xi) = \sigma_\epsilon\Phi_2(\xi) - \Phi_1(\xi) + \epsilon$. Then, by the definition of σ_ϵ, $w_\epsilon(\xi) \geq 0$. Note that $w_\epsilon(\pm\infty) = \epsilon$. This together with $\sigma_\epsilon > 1$ implies that there is $\xi \in \mathbb{R}$ such that $w_\epsilon(\xi) = 0$. Choose a sequence $0 < \epsilon_n \leq \epsilon_0$ for $n = 1, 2...$ such that $\lim\limits_{n\to\infty} \epsilon_n = 0$. Let $\xi_n \in \mathbb{R}$ be such that $w_{\epsilon_n}(\xi_n) = 0$. By $w_{\epsilon_n}(\xi) \geq 0$ for all $\xi \in \mathbb{R}$, $w_{\epsilon_n}(\xi)$ obtains the minimum 0 at ξ_n. This implies that $w_{\epsilon_n}'(\xi_n) = 0$. We claim that $\{\xi_n\} \subset [-(L + L_0), L + L_0]$. Note that $w_{\epsilon_n}(\xi)$ satisfies that

$$cw_{\epsilon_n}'(\xi) + \int_{\mathbb{R}} \kappa(\eta - \xi)w(\eta)d\eta - w(\xi) + \sigma_{\epsilon_n} f(\xi, \Phi_2)\Phi_2(\xi) - f(\xi, \Phi_1)\Phi_1(\xi) = 0.$$

Suppose that $|\xi_n| > L + L_0$ for some n. Then $f(\xi_n, \Phi_2(\xi_n)) = f(\xi_n, \Phi_1(\xi_n)) = -q$. Plugging ξ_n into the above equation, we have that

$$\begin{aligned}
0 &= cw_{\epsilon_n}'(\xi_n) + \int_{\mathbb{R}} \kappa(\eta - \xi_n)w(\eta)d\eta - w(\xi_n) + \sigma_{\epsilon_n} f(\xi_n, \Phi_2)\Phi_2(\xi_n) - f(\xi_n, \Phi_1)\Phi_1(\xi_n) \\
&= \int_{\mathbb{R}} \kappa(\eta - \xi_n)w(\eta)d\eta - q(\sigma_{\epsilon_n}\Phi_2(\xi_n) - \Phi_1(\xi_n)) \\
&= \int_{\mathbb{R}} \kappa(\eta - \xi_n)w(\eta)d\eta - q(w(\xi_n) - \epsilon_n) \\
&= \int_{\mathbb{R}} \kappa(\eta - \xi_n)w(\eta)d\eta + q\epsilon_n \\
&> 0,
\end{aligned}$$

which causes a contradiction. Hence $\{\xi_n\} \subset [-(L+L_0), L+L_0]$. This implies that $\sigma_{\epsilon_n}\Phi_2(\xi_n) - \Phi_1(\xi_n) + \epsilon_n = 0$. Hence $\sigma_{\epsilon_n} = \frac{\Phi_1(\xi_n) - \epsilon_n}{\Phi_2(\xi_n)}$ is bounded and $\sigma^* \in [1, \infty)$. Moreover, there exists a subsequence of ξ_n such that $\lim\limits_{n_k\to\infty} \xi_{n_k}$ exists, denoted by $\lim\limits_{n_k\to\infty} \xi_{n_k} = \xi^*$. Moreover, as n_k goes to infinity, we have that $w^*(\xi) = \sigma^*\Phi_2(\xi) - \Phi_1(\xi)$ with $w^*(\xi^*) = 0$ and ${w^*}'(\xi^*) = 0$. Note that

$$\begin{aligned}
0 &= c{w^*}'(\xi) + \int_{\mathbb{R}} \kappa(\eta - \xi)w^*(\eta)d\eta - w^*(\xi) + \sigma^* f(\xi, \Phi_2)\Phi_2(\xi) - f(\xi, \Phi_1)\Phi_1(\xi) \\
&\geq c{w^*}'(\xi) + \int_{\mathbb{R}} \kappa(\eta - \xi)w^*(\eta)d\eta - w^*(\xi) - qw^*(\xi).
\end{aligned}$$

Plugging $\xi = \xi^*$ into above inequality, $0 \geq \int_{\mathbb{R}} \kappa(\eta - \xi^*)w^*(\eta)d\eta$, which implies that $w^*(\xi) = 0$ for all ξ. Hence, we have that $\Phi_1(\xi) = \sigma^*\Phi_2(\xi)$ and thus

$$\begin{aligned}
0 &= c\Phi_1'(\xi) + \int_{\mathbb{R}} \kappa(\eta - \xi)\Phi_1(\eta)d\eta - \Phi_1(\xi) + f(\xi, \Phi_1)\Phi_1(\xi) \\
&= \sigma^*(c\Phi_2'(\xi) + \int_{\mathbb{R}} \kappa(\eta - \xi)\Phi_2(\eta)d\eta - \Phi_2(\xi) + f(\xi, \sigma^*\Phi_2)\Phi_2(\xi)) \\
&= \sigma^*(c\Phi_2'(\xi) + \int_{\mathbb{R}} \kappa(\eta - \xi)\Phi_2(\eta)d\eta - \Phi_2(\xi) + f(\xi, \Phi_2)\Phi_2(\xi)) \\
&\qquad + (f(\xi, \sigma^*\Phi_2) - f(\xi, \Phi_2))\sigma^*\Phi_2(\xi) \\
&= (f(\xi, \sigma^*\Phi_2) - f(\xi, \Phi_2))\sigma^*\Phi_2(\xi),
\end{aligned}$$

which implies that $f(\xi,\sigma^*\Phi_2)-f(\xi,\Phi_2)=0$. In particular, $f(0,\sigma^*\Phi_2(0))-f(0,\Phi_2(0))=0$, which implies that $r(1-\sigma^*\Phi_2(0))=r(1-\Phi_2(0))$ and then $\sigma^*=1$. □

6. Spectral theory of nonlocal dispersal operators and its applications

In this section, we study the spectral theory of the linearized equation of (1.5) at $v=0$, i.e., (1.10), and discuss its applications to the persistence and extinction in (1.5).

Letting $v(t,\xi)=e^{\lambda t}\psi(\xi)$, (1.10) yields

$$c\psi'(\xi)+\int_{\mathbb{R}}k(\eta-\xi)\psi(\eta)d\eta-\psi(\xi)+f(\xi,0)\psi(\xi)=\lambda\psi(\xi). \tag{6.1}$$

It is obvious that $v(t,\xi)=e^{\lambda t}\psi(\xi)$ is a solution of (1.10) if and only if (λ,ψ) satisfies Eq. (6.1).

Let

$$X^1=\{u\in X \mid u'(\cdot)\in X\}$$

and

$$(\mathcal{L}(c)\phi)(\xi):=c\phi'(\xi)+\int_{\mathbb{R}}k(\eta-\xi)\phi(\eta)d\eta-\phi(\xi)+f(\xi,0)\phi(\xi)$$

for $\phi\in X^1$. Let $\sigma(\mathcal{L}(c))$ be the spectrum of $\mathcal{L}(c)$ acting on X^1.

Definition 6.1. Let

$$\lambda(c,L)=\sup\{\mathrm{Re}\lambda \mid \lambda\in\sigma(\mathcal{L}(c))\}.$$

$\lambda(c,L)$ is called the principal spectral point of $\mathcal{L}(c)$ or (1.10). $\lambda(c,L)$ is called the principal eigenvalue of $\mathcal{L}(c)$ if $\mathcal{L}(c)$ has an eigenfunction in $X^+\setminus\{0\}$ associated with $\lambda(c,L)$.

The objective of this section is to study the properties of $\lambda(c,L)$ and their applications to the persistence and extinction in (1.5). To do so, we first study in the next subsection the properties of the spectrum of (6.1) with $f(\xi,0)$ being replaced by some periodic function.

6.1. Existence of principal eigenvalue with periodic dependence

In this subsection, we shall consider the eigenvalue problem (6.1) with $f(\xi,0)$ being replaced by $a(\xi)$, where $a(\cdot)\in X_p$ and

$$X_p:=\{a(\cdot)\in X|a(\cdot)=a(\cdot+p)\}$$

for $p>0$, that is,

$$c\phi'(\xi)+\int_{\mathbb{R}}k(\eta-\xi)\phi(\eta)d\eta-\phi(\xi)+a(\xi)\phi(\xi)=\lambda\phi(\xi),\quad \phi\in X_p^1, \tag{6.2}$$

where $X_p^1:=\{u\in X|u,u'\in X_p\}$. We denote $X_p^+=\{u\in X_p|u\geq 0\}$ and $Int(X_p^+)=\{u\in X_p|u>0\}$, which is the interior of X_p^+.

For given $a\in X_p$, let

$$(\mathcal{L}(c,a;p)\phi)(\xi):=c\phi'(\xi)+\int_{\mathbb{R}}k(\eta-\xi)\phi(\eta)d\eta-\phi(\xi)+a(\xi)\phi(\xi)$$

for $\phi\in X_p^1$. Let $\sigma(\mathcal{L}(c,a;p))$ be the spectrum of $\mathcal{L}(c,a;p)$ acting on X_p^1, and

$$\lambda_p(c,a)=\sup\{\mathrm{Re}\lambda \mid \lambda\in\sigma(\mathcal{L}(c,a;p))\}.$$

$\lambda_p(c,a)$ is called the *principal spectral point* of $\mathcal{L}(c,a;p)$. $\lambda_p(c,a)$ is called the *principal eigenvalue* of $\mathcal{L}(c,a;p)$ if $\mathcal{L}(c,a;p)$ has an eigenfunction in $X_p^+\setminus\{0\}$ associated with $\lambda_p(c,a)$. We have the following theorem.

Theorem 6.1. *Assume that a is Lipschitz continuous. Then the principal eigenvalue $\lambda_p(c,a)$ of $\mathcal{L}(c,a;p)$ always exists.*

To prove the above theorem, we first prove some lemmas. In the following, if no confusion occurs, we may write $\mathcal{L}(c,a;p)$ as $\mathcal{L}$. Let $\mathcal{K}: X_p \to X_p$ and $\mathcal{T}: X_p^1 \to X_p$ be defined by

$$\mathcal{K}u(\xi) = \int_{\mathbb{R}} k(\eta-\xi)\phi(\eta)d\eta$$

and

$$(\mathcal{T}\phi)(\xi) := c\phi'(\xi) - \phi(\xi) + a(\xi)\phi(\xi).$$

Then we may write $\mathcal{L}$ as $\mathcal{K}+\mathcal{T}$, and write (6.2) as

$$(\mathcal{K}+\mathcal{T})\phi = \lambda\phi.$$

Note that if $(\lambda I - \mathcal{T})^{-1}$ exists, then (6.2) has nontrivial solutions (λ,ϕ) with ϕ in $X_p^1 \setminus \{0\}$ if and only if

$$\mathcal{K}(\lambda I - \mathcal{T})^{-1}v = v \tag{6.3}$$

has nontrivial solutions (λ, v) with $v \in X_p \setminus \{0\}$.

For $c = 0$, Theorem 6.1 has been proved in [4] (see also [19]), that is, $\lambda_p(0,a)$ is the principal eigenvalue of $\mathcal{L}(0,a;p)$ with a periodic principal eigenfunction. In the rest of the subsection, we assume that $c > 0$.

Let

$$\lambda_{\mathcal{T}} = -1 + \bar{a}, \quad \text{where } \bar{a} = \frac{\int_0^p a(s)ds}{p}. \tag{6.4}$$

Lemma 6.1. *Assume that $c > 0$.*

(1) If $\alpha \in \mathbb{C}$ and $Re\alpha > \lambda_{\mathcal{T}}$, then $(\alpha I - \mathcal{T})^{-1}$ exists.
(2) If $\alpha \in \mathbb{R}$ and $\alpha > \lambda_{\mathcal{T}}$, then $\mathcal{K}(\alpha I - \mathcal{T})^{-1}$ is a compact operator on X_p and is strongly positive, i.e., $\mathcal{K}(\alpha I - \mathcal{T})^{-1}u \in Int(X_p^+)$ if $u \in X_p^+ \setminus \{0\}$.

Proof. (1) For given $w \in X_p$, consider $(\alpha I - \mathcal{T})\phi = w$, i.e.

$$\phi'(\xi) - \frac{1}{c}[\alpha + 1 - a(\xi)]\phi(\xi) = -\frac{w(\xi)}{c}. \tag{6.5}$$

If a solution ϕ in X_p^1 exists, then we must have that:

$$\frac{d}{d\xi}\Big[\exp\Big(\frac{1}{c}\int_\xi^0 (\alpha+1-a(\eta))d\eta\Big)\phi(\xi)\Big] = -\exp\Big(\frac{1}{c}\int_\xi^0 (\alpha+1-a(\eta))d\eta\Big)\frac{w(\xi)}{c}.$$

Therefore, we integrate both sides over $[\xi,\infty)$ and exploiting the fact that $\mathrm{Re}(\alpha) > -1+\bar{a}$ and that ϕ must belong to X_p^1, we can simplify the result to get that

$$\phi(\xi) = \frac{1}{c}\int_\xi^\infty \exp\Big(\frac{1}{c}\int_\zeta^\xi (\alpha+1-a(\eta))d\eta\Big)w(\zeta)d\zeta. \tag{6.6}$$

For each ζ in $[\xi,\infty)$, let k be the unique non-negative integer such that $\zeta \in [\xi+kp, \xi+(k+1)p)$. Then

$$\begin{aligned}|\int_\zeta^\xi (\bar{a}-a(\eta))d\eta| &= |\int_\xi^\zeta (\bar{a}-a(\eta))d\eta| = |\int_{\xi+kp}^\zeta (\bar{a}-a(\eta))d\eta| \\ &\le \int_{\xi+kp}^\zeta |\bar{a}-a(\eta)|d\eta \le \int_0^p |\bar{a}-a(\eta)|d\eta \le (|\bar{a}| + \max_{\xi\in[0,p]}|a(\xi)|)p,\end{aligned}$$

and therefore we have that

$$\begin{aligned}\phi(\xi) &= \frac{1}{c}\int_{\xi}^{\infty}\exp\big(\frac{1}{c}\int_{\zeta}^{\xi}(\alpha+1-a(\eta))d\eta\big)w(\zeta)d\zeta \\ &= \frac{1}{c}\int_{\xi}^{\infty}\exp\big(\frac{1}{c}\int_{\zeta}^{\xi}(\alpha+1-\bar{a})d\eta\big)\exp\big(\frac{1}{c}\int_{\zeta}^{\xi}(\bar{a}-a(\eta))d\eta\big)w(\zeta)d\zeta \\ &\le \frac{1}{c}\cdot exp\Big(\frac{1}{c}\big(|\bar{a}|+\max_{\xi\in[0,p]}|a(\xi)|\big)p\Big)\cdot\|w\|_{\infty}\cdot\int_{\xi}^{\infty}\exp\big(\frac{1}{c}\int_{\zeta}^{\xi}(\alpha+1-\bar{a})d\eta\big)d\zeta \\ &= \frac{1}{\alpha+1-\bar{a}}\cdot\exp\Big(\frac{1}{c}\big(|\bar{a}|+\max_{\xi\in[0,p]}|a(\xi)|\big)p\Big)\cdot\|w\|_{\infty}.\end{aligned}$$

Then with $C := \frac{1}{\alpha+1-\bar{a}}\cdot\exp\Big(\frac{1}{c}\big(|\bar{a}|+\max_{\xi\in[0,p]}|a(\xi)|\big)p\Big)$, we have just shown that $\|\phi\|_{\infty}\le C\|w\|_{\infty}$.

Moreover, letting $\hat{\zeta}=\zeta-p$, we have that

$$\begin{aligned}\phi(\xi+p) &= \frac{1}{c}\int_{\xi+p}^{\infty}\exp\big(\frac{1}{c}\int_{\zeta}^{\xi+p}(\alpha+1-a(\eta))d\eta\big)w(\zeta)d\zeta \\ &= \frac{1}{c}\int_{\xi}^{\infty}\exp\big(\frac{1}{c}\int_{\hat{\zeta}+p}^{\xi+p}(\alpha+1-a(\eta))d\eta\big)w(\hat{\zeta}+p)d(\hat{\zeta}+p) \\ &= \frac{1}{c}\int_{\xi}^{\infty}\exp\big(\frac{1}{c}\int_{\hat{\zeta}}^{\xi}(\alpha+1-a(\eta))d\eta\big)w(\hat{\zeta})d\hat{\zeta} \\ &= \phi(\xi).\end{aligned}$$

These arguments establish the existence of a solution ϕ in X_p^1 to Eq. (6.5), for each w in X_p. Uniqueness of ϕ follows from standard results in the theory of linear ODEs. This concludes the proof of part (1) of this Lemma.

(2) This follows from the compactness and the positivity of $\mathcal{K}$ and the strong positivity of $(\alpha I-\mathcal{T})^{-1}$. □

Lemma 6.2. *Assume that $c>0$. Then $\lambda_{\mathcal{T}}$ is an eigenvalue of $\mathcal{T}$ and its associated eigenfunction is $\phi(\xi)=exp\big(\frac{1}{c}\big(\bar{a}\xi-\int_0^{\xi}a(s)ds\big)\big)$.*

Proof. This can be verified by direct computation. □

Lemma 6.3. *Assume that $\alpha>\lambda_{\mathcal{T}}$ and $c>0$. Let $\rho(\alpha)$ be the spectral radius of $\mathcal{K}(\alpha I-\mathcal{T})^{-1}$.*

(1) $\rho(\alpha_1)>\rho(\alpha_2)$ if $\alpha_2>\alpha_1$.
(2) $\rho(\alpha)\to 0$ as $\alpha\to\infty$.
(3) $\rho(\alpha)$ is continuous in $\alpha>\lambda_{\mathcal{T}}$.
(4) For some $\alpha_0>\lambda_{\mathcal{T}}$, if $\rho(\alpha_0)>1$ then there exists a λ such that $\rho(\lambda)=1$.

Proof. (1) By Lemma 6.1, $\mathcal{K}(\alpha I-\mathcal{T})^{-1}$ is a strongly positive and compact operator and then by Theorem 19.3 (Krein–Rutman theorem) in [20], $\rho(\alpha)$ is its principal eigenvalue with a positive eigenvector. Note that for given $w\in X_p$,

$$(\alpha I-\mathcal{T})^{-1}w=\frac{1}{c}\int_{\xi}^{\infty}\exp\big(\frac{1}{c}\int_{\zeta}^{\xi}(\alpha+1-a(\eta))d\eta\big)w(\zeta)d\zeta.$$

Hence for $w\in X_p^+\setminus\{0\}$,

$$(\alpha_1 I-\mathcal{T})^{-1}w>(\alpha_2 I-\mathcal{T})^{-1}w$$

and then

$$\mathcal{K}(\alpha_1 I-\mathcal{T})^{-1}w>\mathcal{K}(\alpha_2 I-\mathcal{T})^{-1}w.$$

By Theorem 19.3 (d) in [20], $\rho(\alpha_1) > \rho(\alpha_2)$.

(2) From the arguments of Lemma 6.1(1), we have that $\lim_{\alpha\to\infty} \|(\alpha I - \mathcal{T})^{-1}w\| = 0$ for any $w \in X_p$. The assertion holds since $\rho(\alpha) \le \|\mathcal{K}\|\|(\alpha I - \mathcal{T})^{-1}\|$.

(3) This follows from Lemma 2 in [21].

(4) This follows from (1), (2), and (3). □

Lemma 6.4. *Assume that $c > 0$. Let $V(t; c, a)$ be the solution operator of (1.10) with $f(\xi, 0)$ being replaced by $a(\xi)$, that is, $v(t, \cdot; v_0) = V(t; c, a)v_0$ is the solution of (1.10) with $f(\xi, 0)$ being replaced by $a(\xi)$ and $v(0, \cdot; v_0) = v_0(\cdot) \in X_p$. Then*

$$\lambda_p(c, a) = \limsup_{t\to\infty} \frac{\ln \|V(t; c, a)\|}{t}.$$

Proof. This follows from similar arguments as those in [22, Proposition 2.5]. For completeness, we provide a sketch of the proof in the following.

First, let $\lambda_L = \limsup_{t\to\infty} \frac{\ln \|V(t;c,a)\|}{t}$. For any given $\tilde{\lambda} > \limsup_{t\to\infty} \frac{\ln \|V(t;c,a)\|}{t}$, there is $M > 0$ such that

$$\|V(t; c, a)\| \le Me^{\tilde{\lambda}t} \quad \forall\, t \ge 0.$$

Then for any $\epsilon > 0$, we have that

$$\|e^{(-\tilde{\lambda}-\epsilon)t}V(t; c, a)\| \le Me^{-\epsilon t} \quad \forall\, t \ge 0. \tag{6.7}$$

Let $\tilde{v} = e^{(-\tilde{\lambda}-\epsilon)t}v$. Then $\tilde{v}$ satisfies

$$\tilde{v}_t = c\tilde{v}_x + \int_{\mathbb{R}} k(y - x)\tilde{v}(t, y)dy - \tilde{v}(t, x) + a(x)\tilde{v}(t, x) - (\tilde{\lambda} + \epsilon)\tilde{v}. \tag{6.8}$$

Let $\tilde{V}(t; c, a)$ be the solution operator of (6.8). For any $w \in X_p$, let

$$\tilde{v}(t, x) = \int_{-\infty}^{t} \tilde{V}(t - \tau; c, a)w(\cdot)d\tau. \tag{6.9}$$

Then, by Leibniz integral rule, for $-\infty < a(x), b(x) < \infty$,

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)\,dt\right) = f\big(x, b(x)\big) \cdot \frac{d}{dx}b(x) - f\big(x, a(x)\big) \cdot \frac{d}{dx}a(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)\,dt,$$

we have that

$$\begin{aligned}\frac{\partial \tilde{v}(t,x)}{\partial t} &= w(x) + \int_{-\infty}^{t} \frac{\partial}{\partial t}\tilde{V}(t - \tau; c, a)w(\cdot)d\tau \\ &= c\tilde{v}_x + \int_{\mathbb{R}} k(y - x)\tilde{v}(t, y)dy - \tilde{v}(t, x) + a(x)\tilde{v}(t, x) - (\tilde{\lambda} + \epsilon)\tilde{v} + w(x).\end{aligned}$$

Thus it is a solution of

$$\tilde{v}_t = c\tilde{v}_x + \int_{\mathbb{R}} k(y - x)\tilde{v}(t, y)dy - \tilde{v}(t, x) + a(x)\tilde{v}(t, x) - (\tilde{\lambda} + \epsilon)\tilde{v} + w(x) \tag{6.10}$$

on $t \in \mathbb{R}$. Letting $t \to \infty$ in (6.9),

$$\lim_{t\to\infty} \tilde{v}(t, x) = \int_{0}^{\infty} \tilde{V}(\tau; c, a)w(\cdot)d\tau =: \tilde{v}(x; w) \in X_p.$$

Moreover,

$$\|\tilde{v}(\cdot;w)\| \leq \frac{M}{\epsilon}\|w\|.$$

Suppose that $\tilde{v}_1(x;w)$ and $\tilde{v}_2(x;w)$ are two stationary solutions of (6.10) in X_p and then $\tilde{v}_1(x;w) - \tilde{v}_2(x;w)$ is a stationary solution of Eq. (6.8). The estimate (6.7) implies that Eq. (6.8) has only trivial stationary solution and so $\tilde{v}_1(x;w) = \tilde{v}_2(x;w)$, that is, $\tilde{v}(x;w)$ is the unique stationary solution of (6.10). This implies that $(\mathcal{K}+\mathcal{T}-(\tilde{\lambda}+\epsilon)I)^{-1}$ exists for any $\epsilon > 0$ and so $(\tilde{\lambda},\infty)$ is in the resolvent of the operator $\mathcal{K}+\mathcal{T}$. While $\lambda_p(c,a) = \sup\{\mathrm{Re}\lambda \mid \lambda \in \sigma(\mathcal{K}+\mathcal{T})\}$, we have that $\tilde{\lambda}+\epsilon > \lambda_p(c,a)$. Hence, as $\epsilon > 0$ was arbitrary,

$$\lambda_p(c,a) \leq \limsup_{t\to\infty} \frac{\ln\|V(t;c,a)\|}{t}.$$

Next, for any $\epsilon > 0$ and $M > 0$, let $\bar{\lambda} = \lambda_p(c,a) + \epsilon$ and $v_M(x)$ be the unique solution of

$$cv_x + \int_{\mathbb{R}} k(y-x)v(y)dy - v(x) + a(x)v - \bar{\lambda}v(x) + M = 0, \quad x \in \mathbb{R}. \tag{6.11}$$

Then with $\tilde{\lambda} = \frac{1+q+|\bar{\lambda}|}{c} > 0$ and $\tilde{f}(x,v) = \frac{1}{c}(\int_{\mathbb{R}} k(y-x)v(y)dy + (q+a(x))v(x) + (|\bar{\lambda}|-\bar{\lambda})v(x) + M) \geq \frac{M}{c}$, Eq. (6.11) is equivalent to that:

$$cv_x + \int_{\mathbb{R}} k(y-x)v(y)dy - v(x) - qv(x) - |\bar{\lambda}|v(x) + (q+a(x))v + (|\bar{\lambda}|-\bar{\lambda})v(x) + M = 0, \quad x \in \mathbb{R},$$

and thus

$$\tilde{\lambda}v - v' = \tilde{f}(x,v), \quad x \in \mathbb{R}. \tag{6.12}$$

Then multiply (6.12) by $e^{-\tilde{\lambda}x}$ to get that $[-e^{-\tilde{\lambda}x}v(x)]' = e^{-\tilde{\lambda}x}\tilde{f}(x,v(x))$, and integrate both sides over $[x,\infty)$ to

$$v(x) = e^{\tilde{\lambda}x}\int_x^\infty (e^{-\tilde{\lambda}s}\tilde{f}(s,v(s)))ds \geq e^{\tilde{\lambda}x}\int_x^\infty (e^{-\tilde{\lambda}s}\frac{M}{c})ds = \frac{M}{c\tilde{\lambda}}, \quad x \in \mathbb{R}.$$

Choose $M \geq |\bar{\lambda}| + 1 + q$ and then we have that $v_M(x) \geq 1$.

Note that $v_M(x)$ is a super-solution of (6.8). By the comparison principle (Proposition 2.1) for (6.8), we have that

$$0 < e^{-\bar{\lambda}t}V(t;c,a)\cdot 1 \leq v_M(x), \quad \forall t \geq 0,\ x \in \mathbb{R}.$$

This implies that

$$\limsup_{t\to\infty} \frac{\ln\|V(t;c,a)\|}{t} \leq \limsup_{t\to\infty} \frac{\ln\|V(t;c,a)\cdot 1\|}{t} \leq \bar{\lambda},$$

for all $\epsilon > 0$, and thus also that $\limsup_{t\to\infty} \frac{\ln\|V(t;c,a)\|}{t} \leq \lambda_p(c,a)$. This concludes the proof of the lemma. □

We now prove Theorem 6.1.

Proof of Theorem 6.1. For $c = 0$, it has been proved in [4]. Now we assume that $c > 0$.

Suppose that $\lambda > \lambda_{\mathcal{T}}$ and let $(\lambda_{\mathcal{T}},\phi)$ be as in Lemma 6.2 such that

$$c\phi'(\xi) - \phi(\xi) + a(\xi)\phi(\xi) = \lambda_{\mathcal{T}}\phi(\xi),$$

and then

$$-c\phi'(\xi) + (\lambda + 1 - a(\xi))\phi(\xi) = (\lambda - \lambda_{\mathcal{T}})\phi(\xi),$$

denoted by

$$(-c\partial_\xi + (\lambda + 1 - a(\cdot))I)\phi = (\lambda - \lambda_{\mathcal{T}})\phi.$$

Hence

$$(-c\partial_\xi + (\lambda + 1 - a(\cdot))I)^{-1}(\lambda - \lambda_{\mathcal{T}})\phi = \phi.$$

This implies that

$$\mathcal{K}(-c\partial_\eta + (\lambda + 1 - a(\cdot))I)^{-1}(\lambda - \lambda_{\mathcal{T}})\phi = \mathcal{K}\phi > (\lambda - \lambda_{\mathcal{T}})\phi$$

for $0 < \lambda - \lambda_{\mathcal{T}} \ll 1$. It then follows that

$$\rho(\mathcal{K}(-c\partial_\eta + (\lambda + 1 - a(\cdot))I)^{-1}) > 1$$

for $0 < \lambda - \lambda_{\mathcal{T}} \ll 1$.

By Lemma 6.3

$$\rho(\mathcal{K}(-c\partial_\eta + (\lambda + 1 - a(\cdot))I)^{-1}) \to 0$$

as $\lambda \to \infty$. Hence there are $\hat{\lambda} > \lambda_{\mathcal{T}}$ and a p-periodic positive function ψ such that

$$\rho(\mathcal{K}(-c\partial_\eta + (\hat{\lambda} + 1 - a(\cdot))I)^{-1}) = 1.$$

$$\int_{-\infty}^{\infty} \kappa(\eta - \xi)\big(-c\partial_\eta + (\hat{\lambda} + 1 - a(\cdot))I\big)^{-1}\psi(\eta)d\eta = \psi(\eta). \tag{6.13}$$

and thus

$$c\tilde{\phi}'(\xi) + \int_{-\infty}^{\infty} \kappa(\eta - \xi)\tilde{\phi}(\eta)d\eta - \tilde{\phi}(\xi) + a(\xi)\tilde{\phi}(\xi) = \hat{\lambda}\tilde{\phi}(\xi), \tag{6.14}$$

where $\tilde{\phi}(\xi) = \big(-c\partial_\xi + (\hat{\lambda} + 1 - a(\cdot))I\big)^{-1}\psi(\xi)$. Note that $\tilde{\phi}(\xi) > 0$ on $\mathbb{R}$ because $\psi(\xi) > 0$ on $\mathbb{R}$, and since $\big(-c\partial_\xi + (\hat{\lambda} + 1 - a(\cdot))I\big)^{-1}$ is strongly positive (see Lemma 6.1).Therefore, $\hat{\lambda} \in \sigma(\mathcal{L})$.

Next, we show that $\hat{\lambda} = \lambda_p(c, a)$. Let $V(t; c, a)$ be as in Lemma 6.4, the solution operator of (1.10) with $f(\xi, 0)$ being replaced by $a(\xi)$, We have that

$$V(t; c, a)\tilde{\phi} = e^{\hat{\lambda}t}\tilde{\phi} \quad \forall\, t > 0.$$

For any $\phi \in Int(X_p^+)$ with $\|\phi\|_\infty = 1$, there exist positive σ_1 and σ_2 such that $\sigma_1\tilde{\phi} \le \phi \le \sigma_2\tilde{\phi}$. By comparison principle (Proposition 2.1), $\sigma_1 V(t; c, a)\tilde{\phi} \le V(t; c, a)\phi \le \sigma_2 V(t; c, a)\tilde{\phi}$ and thus $\sigma_1\|V(t; c, a)\tilde{\phi}\| \le \|V(t; c, a)\phi\| \le \sigma_2\|V(t; c, a)\tilde{\phi}\|$. Since ϕ is arbitrary and $\|\phi\|_\infty = 1$, $\sigma_1\|V(t; c, a)\tilde{\phi}\| \le \|V(t; c, a)\| \le \sigma_2\|V(t; c, a)\tilde{\phi}\|$. Therefore, $\frac{ln(\sigma_1\|V(t;c,a)\tilde{\phi}\|)}{t} \le \frac{ln(\|V(t;c,a)\|)}{t} \le \frac{ln(\sigma_2\|V(t;c,a)\tilde{\phi}\|)}{t}$ for $t > 0$. Then $\frac{ln(\sigma_1\|e^{\hat{\lambda}t}\tilde{\phi}\|)}{t} \le \frac{ln(\|V(t;c,a)\|)}{t} \le \frac{ln(\sigma_2\|e^{\hat{\lambda}t}\tilde{\phi}\|)}{t}$, and thus $\hat{\lambda} + \frac{ln(\sigma_1\|\tilde{\phi}\|)}{t} \le \frac{ln(\|V(t;c,a)\|)}{t} \le \hat{\lambda} + \frac{ln(\sigma_2\|\tilde{\phi}\|)}{t}$. Letting $t \to \infty$, we have that

$$\limsup_{t\to\infty} \frac{\ln\|V(t; c, a)\|}{t} = \hat{\lambda}$$

and hence $\hat{\lambda} = \lambda_p(c, a)$ by Lemma 6.4 . □

The following lemma shows the dependence of $\lambda_p(c, a)$ on $a(\xi)$.

Lemma 6.5. $\lambda_p(c, a_1) \le \lambda_p(c, a_2)$ *whenever* $a_1(\xi) \le a_2(\xi)$. *Moreover,* $\lambda_p(c, a_1) < \lambda_p(c, a_2)$ *if* $a_1(\xi) \le a_2(\xi)$ *and* $a_1(\xi) \ne a_2(\xi)$.

Proof. With Lemma 6.4 and the comparison principle (Proposition 2.1), we have that

$$\lambda_p(c, a_1) = \limsup_{t\to\infty} \frac{\ln\|V(t; c, a_1)\|}{t} \le \limsup_{t\to\infty} \frac{\ln\|V(t; c, a_2)\|}{t} = \lambda_p(c, a_2).$$

We shall prove the second statement by contradiction. Suppose that $\lambda_p(c,a_1)=\lambda_p(c,a_2):=\bar\lambda_p$ and $\phi_i(\xi)$ is the corresponding positive principal eigenfunction to $\lambda_p(c,a_i)$ for $i=1,2$, that is,

$$c\phi_i'(\xi)+\int_{-\infty}^{\infty}\kappa(\eta-\xi)\phi_i(\eta)d\eta-\phi_i(\xi)+a_i(\xi)\phi(\xi)=\bar\lambda_p\phi_i(\xi).$$

Then

$$\int_{-\infty}^{\infty}\kappa(\eta-\xi)\phi_i(\eta)d\eta=-c\phi_i'(\xi)+\phi_i(\xi)-a_i(\xi)\phi(\xi)+\bar\lambda_p\phi_i(\xi),$$

denoted by $\mathcal{K}\phi_i=(-c\partial_\xi+(\bar\lambda_p+1-a_i(\cdot))I)\phi_i$ for $i=1,2$. Hence we have that $\mathcal{K}(-c\partial_\eta+(\bar\lambda_p+1-a_i(\cdot))I)^{-1}w=w$ with $w(\xi)=-c\phi_i'(\xi)+\phi_i(\xi)-a_i(\xi)\phi(\xi)+\bar\lambda_p\phi_i(\xi)=\int_{-\infty}^{\infty}\kappa(\eta-\xi)\phi_i(\eta)d\eta>0$ for $i=1,2$. This implies that $\rho(\mathcal{K}(-c\partial_\eta+(\bar\lambda_p+1-a_i(\cdot))I)^{-1})=1$ for $i=1,2$. On the other hand, by the arguments in the proof of Lemma 6.3(1), we have that $\rho(\mathcal{K}(-c\partial_\eta+(\bar\lambda_p+1-a_1(\cdot))I)^{-1})>\rho(\mathcal{K}(-c\partial_\eta+(\bar\lambda_p+1-a_2(\cdot))I)^{-1})$ if $a_1(\xi)\le a_2(\xi)$ and $a_1(\xi)\ne a_2(\xi)$, which is a contradiction. □

6.2. Dependence of principal eigenvalue on moving speed c and patch size L

In this subsection, we explore some important properties of $\lambda(c,L)$. Recall that $\lambda(c,L)$ is the principal spectrum point of the spectral problem (6.1), that is, the spectral problem associated to the linearization of (1.5) at the trivial solution $v\equiv 0$. In particular, we study the dependence of $\lambda(c,L)$ on c and L.

The main results of this subsection are stated in the following theorem.

Theorem 6.2.

(1) $\lambda(c,L)$ is a principal eigenvalue. Moreover, let $\phi(\xi)$ be a corresponding positive eigenfunction, then there are $\tilde M_\pm$ such that

$$\limsup_{\xi\to\infty}\frac{\phi(\xi)}{e^{\mu_-(\lambda(c,L))\xi}}\le\tilde M_+ \tag{6.15}$$

and

$$\limsup_{\xi\to-\infty}\frac{\phi(\xi)}{e^{\mu_+(\lambda(c,L))\xi}}\le\tilde M_-, \tag{6.16}$$

where $\mu_\pm(\lambda)$ is defined in (1.9).

(2) $\lambda(c,L)$ is continuous in $(c,L)\in(0,\infty)\times(0,\infty)$.

(3) For any fixed $c>0$, $\lambda(c,L)$ is strictly increasing in $L>0$.

(4) If $0<c<c^*$, there is $0\le L^{**}<\infty$ such that $\lambda(c,L)>0$ for all $L>L^{**}$, and for any $0<L<L^{**}$, $\lambda(c,L)\le 0$.

(5) If $c>c^*$, then $\lambda(c,L)<0$.

To prove the above Theorem we shall first prove some auxiliary results. Pick $\frac{p}{2}>L_0+L$ and define a p-periodic function $a_p(\xi;L,L_0)$ on one period as follows:

$$a_p(\xi;L,L_0)=f(\xi,0),\xi\in[-\frac{p}{2},\frac{p}{2}].$$

Observe that

$$a_p(\xi;L,L_0)\ge a_{2p}(\xi;L,L_0)\ge\cdots\ge a_{2np}(\xi;L,L_0)\ge\cdots\ge f(\xi,0)\ge -q,$$

and then with $a_{2np}(0)=r>-q$, Lemma 6.5 implies that

$$\lambda_p(c,a_p)\ge\lambda_p(c,a_{2p})\ge\cdots\ge\lambda_p(c,a_{2np})\ge\cdots>\lambda_p(c,-q)=-q.$$

Then letting $p\to\infty$, the limit of $\lambda_p(c,a_{2np})$ exists, and let

$$\lambda_\infty(c,L,L_0)=\lim_{n\to\infty}\lambda_p(c,a_{2np}).$$

Proposition 6.1. *Let $V(t;c,L)$ be the solution operator of (1.10), that is, $v(t,\cdot;v_0) = V(t;c,L)v_0$ is the solution of (1.10) with $v(0,\cdot;v_0) = v_0(\cdot) \in X$. Then*

$$\lambda(c,L) = \limsup_{t\to\infty} \frac{\ln \|V(t;c,L)\|}{t}.$$

Proof. It can be proved by similar arguments as those in Lemma 6.4. □

Remark 6.1. Let $a(\cdot) \in X$ and $\lambda(a)$ be the principal spectral point of the eigenvalue problem (6.1) on X with $f(\cdot,0)$ being replaced by $a(\cdot)$. Similar to Lemma 6.5, we have that

$$\lambda(a_1) \le \lambda(a_2)$$

for $a_1, a_2 \in X$ with $a_1(x) \le a_2(x)$ $(x \in \mathbb{R})$. By observation, 1 is a positive principal eigenvector of $\lambda(-q)$ and $\lambda(-q) = -q$. Since $f(\xi,0) \ge -q$, we have that

$$\lambda(c,L) \ge \lambda(-q) = -q.$$

Proposition 6.2. *$\lambda(c,L) = \lambda_\infty(c,L,L_0)$ and $\lambda(c,L)$ is an eigenvalue.*

Proof. First, let $\phi_{2np}(\xi)$ be the positive 2np-periodic eigenfunction corresponding to $\lambda_p(c,a_{2np})$ with $\|\phi_{2np}\|_\infty = 1$, that is, $(\lambda_p(c,a_{2np}),\phi_{2np})$ satisfies that

$$c\phi'_{2np}(\xi) + \int_{\mathbb{R}} \kappa(\xi-\eta)\phi_{2np}(\eta)d\eta - \phi_{2np}(\xi) + a_{2np}(\xi;L,L_0)\phi_{2np}(\xi) = \lambda_p(c,a_{2np})\phi_{2np}(\xi), \tag{6.17}$$

Let $\xi_{2np} \in [-np,np]$ be such that $\phi_{2np}(\xi_{2np}) = \sup_{\xi\in\mathbb{R}} \phi_{2np}(\xi) = 1$. Plugging $\phi_{2np}(\xi_{2np}) = 1$ and $\phi'_{2np}(\xi_{2np}) = 0$ into Eq. (6.17), we have that

$$\lambda_p(c,a_{2np}) = \int_{\mathbb{R}} \kappa(\xi_{2np}-\eta)\phi_{2np}(\eta)d\eta - 1 + a_{2np}(\xi_{2np};L,L_0).$$

Note that $\phi_{2np} \le 1$ but not identical to 1 and so $\int_{\mathbb{R}} \kappa(\xi-\eta)\phi_{2np}(\eta)d\eta < 1$ for any $\xi \in \mathbb{R}$. Then $\lambda_p(c,a_{2np}) < a_{2np}(\xi_{2np};L,L_0)$. On the other hand, with Lemma 6.5, we have that $-q < \lambda_p(c,a_{2np})$ and so $-q < a_{2np}(\xi_{2np};L,L_0)$. This implies that $\xi_{2np} \in (-L-L_0, L+L_0) \subset [-np,np]$.

Next, recall that $-q < \lambda_p(c,a_{2np}) < a_{2np}(\xi_{2np};L,L_0)$. Then by Eq. (6.17), we have that

$$\begin{aligned} c|\phi'_{2np}(\xi)| &= |-\int_{\mathbb{R}} \kappa(\xi-\eta)\phi_{2np}(\eta)d\eta + \phi_{2np}(\xi)(-\lambda_p(c,a_{2np}) + 1 - a_{2np}(\xi;L,L_0))| \\ &\le \int_{\mathbb{R}} \kappa(\eta)\|\phi_{2np}\|_\infty d\eta + \|\phi_{2np}\|_\infty(|\lambda_p(c,a_{2np})| + 1 + \|a_{2np}\|_\infty) \\ &\le 2\|\phi_{2np}\|_\infty(1 + \|a_{2np}\|_\infty). \end{aligned}$$

Thus with $\|a_{2np}\|_\infty = \max_{\xi\in\mathbb{R}} |a_{2np}(\xi;L,L_0)| = \max\{r,q\}$ and $\|\phi_{2np}\|_\infty = 1$, we have that

$$\sup_{\xi\in\mathbb{R}} |\phi'_{2np}(\xi)| \le \frac{2}{c}\Big(1 + \max\{r,q\}\Big).$$

Therefore there is $n_k \to \infty$ such that $\xi_{2n_kp} \to \xi^\infty \in [-L-L_0, L+L_0]$ and $\phi_{2n_kp}(\xi) \to \phi_\infty(\xi)$ locally uniformly in $\xi \in \mathbb{R}$. Moreover, we have that $\partial_\xi\phi_\infty(\xi)$ exists and

$$c\partial_\xi\phi_\infty(\xi) + \int_{\mathbb{R}} \kappa(\xi-\eta)\phi_\infty(\eta)d\eta - \phi_\infty(\xi) + f(\xi,0)\phi_\infty(\xi) = \lambda_\infty(c,L,L_0)\phi_\infty(\xi).$$

Since $\phi_\infty(\xi_\infty) = 1$, we have that $\phi_\infty(\xi) \not\equiv 0$. By the arguments as in item (2) of Proposition 2.1, we have that $\phi_\infty(\xi) > 0$ for all $\xi \in \mathbb{R}$. Therefore

$$\lambda_\infty(c, L, L_0) \le \lambda(c, L).$$

Now, since $f(\xi, 0) \le a_{2np}(\xi)$, by Remark 6.1, we have that $\lambda(c, L) \le \lambda(c, a_{2np})$ for all $n \ge 1$. This implies that

$$\lambda(c, L) \le \lambda_\infty(c, L, L_0).$$

The proposition then follows. □

We now prove Theorem 6.2.

Proof of Theorem 6.2. (1) By Proposition 6.2, $\lambda(c, L)$ is a principal eigenvalue. Let $\phi(\xi)$ be a corresponding positive eigenfunction. Recall that $\lambda(c, L) > -q$. (6.15) and (6.16) can be proved by similar arguments as in the proof of Theorem 3.1. (1) thus follows.

(2) By (1) again, $\lambda(c, L)$ is a principal eigenvalue. Let $\phi(\xi; c, L) > 0$ be such that $\sup_{\xi \in \mathbb{R}} \phi(\xi; c, L) = 1$ and

$$c\phi'(\xi; c, L) + \int_{\mathbb{R}} \kappa(\xi - \eta)\phi(\eta; c, L)d\eta - \phi(\xi; c, L) + f(\xi, 0)\phi(\xi; c, L) = \lambda(c, L)\phi(\xi; c, L) \tag{6.18}$$

By (6.15) and (6.16), there is $\xi(c, L) \in \mathbb{R}$ such that

$$\phi(\xi(c, L); c, L) = 1. \tag{6.19}$$

Suppose that $c_m \to c > 0$ and $L_m \to L > 0$ as $m \to \infty$. By similar arguments as those in the proof of Proposition 6.2, $\xi(c_m, L_m) \in (-L_m - L_0, L_m + L_0)$. Without loss of generality, we may assume that $\xi(c_m, L_m) \to \xi_0 \in [-L - L_0, L + L_0]$, $\lambda(c_m, L_m) \to \lambda_0$, and $\phi(\xi; c_m, L_m) \to \phi_0(\xi)$ locally uniformly in $\xi \in \mathbb{R}$. It then follows from (6.18) that

$$c\phi_0'(\xi) + \int_{\mathbb{R}} \kappa(\xi - \eta)\phi_0(\eta)d\eta - \phi_0(\xi) + f(\xi, 0)\phi_0(\xi) = \lambda_0\phi_0(\xi). \tag{6.20}$$

Note that $\phi_0(\xi_0) = 1$. Hence $\phi_0(\xi) > 0$ for all $\xi \in \mathbb{R}$. This implies that

$$\lambda_0 \le \lambda(c, L).$$

On the other hand, let $\phi(\xi) > 0$ be such that $\sup_{\xi \in \mathbb{R}} \phi(\xi) = 1$ and

$$c\phi'(\xi) + \int_{\mathbb{R}} \kappa(\xi - \eta)\phi(\eta)d\eta - \phi(\xi) + f(\xi, 0)\phi(\xi) = \lambda(c, L)\phi(\xi). \tag{6.21}$$

By (6.15) and (6.16) again, for any $\epsilon > 0$, there is $\sigma > 0$ such that

$$\sigma\phi_0(\xi) \ge \phi(\xi) - \epsilon \quad \forall \xi \in \mathbb{R}.$$

Let

$$\sigma_\epsilon = \inf\{\sigma > 0 \mid \sigma\phi_0(\xi) \ge \phi(\xi) - \epsilon \quad \forall \xi \in \mathbb{R}\}.$$

It is clear that σ_ϵ is non-increasing in ϵ and is bounded below by 1. Let $w_\epsilon(\xi) = \sigma_\epsilon\phi_0(\xi) - \phi(\xi) + \epsilon$. Then

$$cw_\epsilon'(\xi) + \int_{\mathbb{R}} \kappa(\eta - \xi)w_\epsilon(\eta)d\eta - w_\epsilon(\xi) + f(\xi, 0)w_\epsilon(\xi) = \lambda_0 w_\epsilon(\xi) + (\lambda_0 - \lambda(c, L))\phi(\xi) + (f(\xi, 0) - \lambda_0)\epsilon.$$

It can be proved by similar arguments as those in the proof of Theorem 5.2 that $\inf_{\xi\in\mathbb{R}} w_\epsilon(\xi) = \min_{\xi\in[-L-L_0,L+L_0]} w_\epsilon(\xi) = 0$, and that σ_ϵ is bounded. Let

$$\sigma^* = \lim_{\epsilon\to 0} \sigma_\epsilon \in [1,\infty).$$

We have

$$\sigma^*\phi_0(\xi) \geq \phi(\xi) \quad \forall \xi \in \mathbb{R}$$

and $\sigma^* > 0$. This implies that

$$V(t,c,L)\phi(\cdot) = e^{\lambda(c,L)t}\phi(\cdot) \leq V(t,c,L)\sigma^*\phi_0(\cdot) = e^{\lambda_0 t}\sigma^*\phi_0(\cdot).$$

Hence, we must have

$$\lambda(c,L) \leq \lambda_0.$$

Therefore, $\lambda(c,L) = \lambda_0$ and $\lambda(c,L)$ is continuous in $(c,L) \in (0,\infty)\times(0,L)$.

(3) Fix $c > 0$ and $0 < L_1 < L_2$. To denote the dependence of $f(\xi,0)$ on L, we denote it by $f_L(\xi)$. Let $\phi_i(\xi) > 0\,(i=1,2)$ be such that $\sup_{\xi\in\mathbb{R}} \phi_i(\xi) = 1$ and

$$c\phi_i'(\xi) + \int_{\mathbb{R}} \kappa(\xi-\eta)\phi_i(\eta)d\eta - \phi_i(\xi) + f_{L_i}(\xi)\phi_i(\xi) = \lambda(c,L_i)\phi_i(\xi) \tag{6.22}$$

$(i=1,2)$. Assume by contradiction that $\lambda(c,L_1) = \lambda(c,L_2) = \lambda(c)$. For given $\epsilon > 0$, let

$$\sigma_\epsilon = \inf\{\sigma > 0 \mid \sigma\phi_1(\xi) \geq \phi_2(\xi) - \epsilon \quad \forall \xi \in \mathbb{R}\}.$$

Let

$$w_\epsilon(\xi) = \sigma_\epsilon\phi_1(\xi) - \phi_2(\xi) + \epsilon.$$

Then we have

$$\begin{aligned} &cw_\epsilon'(\xi) + \int_{\mathbb{R}} \kappa(\eta-\xi)w_\epsilon(\eta)d\eta - w_\epsilon(\xi) + f_{L_1}(\xi)w_\epsilon(\xi) \\ &= \lambda(c)w_\epsilon(\xi) + (f_{L_1}(\xi) - f_{L_2}(\xi))\phi_2(\xi) + (f_{L_1}(\xi) - \lambda(c))\epsilon. \end{aligned}$$

Again, we have $\inf_{\xi\in\mathbb{R}} w_\epsilon(\xi) = \min_{\xi\in[-L_1-L_0,L_1+L_0]} w_\epsilon(\xi) = 0$ and σ_ϵ is bounded and nonincreasing. Let

$$\sigma_* = \lim_{\epsilon\to 0}\sigma_\epsilon$$

and

$$w_*(\xi) = \lim_{\epsilon\to 0} w_\epsilon(\xi).$$

We have $w_*(\xi) \geq 0$, $\min_{\xi\in[-L_1-L_0,L_1+L_0]} w_*(\xi) = 0$, and

$$cw_*'(\xi) + \int_{\mathbb{R}} \kappa(\eta-\xi)w_*(\eta)d\eta - w_*(\xi) + f_{L_1}(\xi)w_*(\xi) = \lambda(c)w_*(\xi) + (f_{L_1}(\xi) - f_{L_2}(\xi))\phi_2(\xi).$$

Let $\xi^* \in [-L_1-L_0, L_1+L_0]$ be such that $w_*(\xi^*) = 0$. Then we have

$$0 \leq \int_{\mathbb{R}} \kappa(\eta-\xi^*)w_*(\eta)d\eta = (f_{L_1}(\xi^*) - f_{L_2}(\xi^*))\phi_2(\xi^*) \leq 0.$$

This together with (H1) implies that there is $l^* > 0$ such that

$$w^*(\xi) = 0 \quad \text{for} \quad |\xi - \xi^*| \leq l^*.$$

By the above arguments with ξ^* being replaced by $\xi \in [\xi^* - l^*, \xi^* + l^*]$, we obtain

$$0 \le \int_{\mathbb{R}} \kappa(\eta - \xi) w_*(\eta) d\eta = (f_{L_1}(\xi) - f_{L_2}(\xi))\phi_2(\xi) \le 0 \quad \text{for} \quad |\xi - \xi^*| \le l^*$$

and then

$$w^*(\xi) = 0 \quad \text{for} \quad |\xi - \xi^*| \le 2l^*.$$

Repeating this processes, we have

$$0 \le \int_{\mathbb{R}} \kappa(\eta - \xi) w_*(\eta) d\eta = (f_{L_1}(\xi) - f_{L_2}(\xi))\phi_2(\xi) \quad \forall \xi \in \mathbb{R},$$

which is a contradiction. Hence $\lambda(c, L)$ is strictly increasing in $L > 0$.

(4) By (3), $\lambda(c, L)$ is increasing in L. Hence there is $0 \le L^{**} \le \infty$ such that $\lambda(c, L) > 0$ for $L > L^{**}$ and $\lambda(c, L) \le 0$ for $L < L^{**}$. It suffices to show that there exists an $L > 0$ such that $\lambda(c, L) > 0$.

To this end, first, for $0 < c < c^*$, take $c' \in (c, c^*)$ and fix it. Consider (1.7) with $r(1-u)u$ being replaced by $r(1-\epsilon-u)u$ for some $0 < \epsilon \ll 1$. For given $u_0 \in X^+$ with nonempty and compact support $\text{supp}(u_0)$, by Proposition 2.4,

$$\liminf_{t \to \infty} \inf_{|x| \le c't} (u_\infty(t, x; u_0) - (1 - \epsilon)) = 0,$$

where $u_\infty(t, x; u_0)$ is the solution of (1.7) with $r(1-u)u$ being replaced by $r(1-\epsilon-u)u$ and $u_\infty(0, x; u_0) = u_0(x)$. Then we have that

$$\liminf_{t \to \infty} \inf_{-(c'+c)t \le x \le (c'-c)t} (u_\infty(t, x + ct; u_0) - (1 - \epsilon)) = 0.$$

Next, it was proved in Theorem 5.1 that $u_\infty(t, x; u_0)$ is C^1 in x if $u_0 \in X^1$, and $v_\infty(t, x; u_0) := u_\infty(t, x + ct; u_0)$ is the solution of

$$v_t = cv_x + \int_{\mathbb{R}} \kappa(y - x) v(t, y) dy - v(t, x) + r(1 - \epsilon - v)v, \quad x \in \mathbb{R} \tag{6.23}$$

with $v_\infty(0, x; u_0) = u_0(x)$ and satisfies

$$\liminf_{t \to \infty} \inf_{-(c'+c)t \le x \le (c'-c)t} (v_\infty(t, x; u_0) - (1 - \epsilon)) = 0.$$

Now choose u_0 such that $u_0 \le (1 - 3\epsilon)/2$. Then there is $T_0 > 0$ such that for any $T > T_0$,

$$v_\infty(T, x; u_0) > (1 - \epsilon)/2, \quad x \in \text{supp}(u_0). \tag{6.24}$$

Let $\tilde{v}_\infty(t, x; u_0)$ be the solution of

$$v_t = cv_x + \int_{\mathbb{R}} \kappa(y - x) v(t, y) dy - v(t, x) + r(1 - \epsilon)v, \quad x \in \mathbb{R}. \tag{6.25}$$

Note that $\tilde{v}_\infty(t, x; u_0)$ is a solution and thus also a super-solution of (6.25), while $v_\infty(t, x; u_0)$ is a sub-solution of (6.25) because

$$\begin{aligned}
&\frac{\partial v_\infty}{\partial t} - (c\frac{\partial v_\infty}{\partial x} + \int_{\mathbb{R}} \kappa(y - x) v_\infty(t, y; u_0) dy - v_\infty(t, x) + r(1 - \epsilon) v_\infty) \\
&= \frac{\partial v_\infty}{\partial t} - (c\frac{\partial v_\infty}{\partial x} + \int_{\mathbb{R}} \kappa(y - x) v_\infty(t, y; u_0) dy - v_\infty(t, x; u_0) + r(1 - \epsilon - v_\infty) v_\infty) \\
&\qquad + r(1 - \epsilon - v_\infty) v_\infty - r(1 - \epsilon) v_\infty \\
&= r(1 - \epsilon - v_\infty) v_\infty - r(1 - \epsilon) v_\infty \\
&= -r v_\infty^2 \\
&\le 0.
\end{aligned}$$

With (6.24) and $\tilde{v}_\infty(0,x;u_0)=v_\infty(0,x;u_0)=u_0$, by the comparison principle (Proposition 2.1) for (6.25), we have that

$$\tilde{v}_\infty(T,x;u_0)\geq v_\infty(T,x;u_0)\geq(1-\epsilon)/2,\quad \forall x\in\operatorname{supp}(u_0),T>T_0. \tag{6.26}$$

Let $v(t,x;u_0,L)$ be the solution of (1.10) with $v(0,x;u_0,L)=u_0(x)$. Replacing $f(\xi,0)$ in (1.10) by $f(\xi,0)-r\epsilon$, we have that

$$\frac{\partial v(t,\xi)}{\partial t}=c\frac{\partial v(t,\xi)}{\partial\xi}+\int_{\mathbb{R}}\kappa(\eta-\xi)v(t,\eta)d\eta-v(t,\xi)+(f(\xi,0)-r\epsilon)v(t,\xi),\quad \xi\in\mathbb{R}. \tag{6.27}$$

Then (6.27) has a solution $e^{-r\epsilon t}v(t,x;u_0,L)$. Apply Proposition 2.3 with replacing (1.10) by (6.27) and (2.4) by (6.25) and get that

$$\lim_{L\to\infty}e^{-r\epsilon t}v(t,x;u_0,L)=\tilde{v}_\infty(t,x;u_0)$$

locally uniformly in $(t,x)\in\mathbb{R}^+\times\mathbb{R}$. Then there exists a large enough L such that

$$e^{-r\epsilon T}v(T,x;u_0,L)\geq\tilde{v}_\infty(T,x;u_0)-\epsilon,\quad \forall x\in\operatorname{supp}(u_0),T>T_0.$$

Thus, with (6.26) and u_0 chosen such that $u_0\leq(1-3\epsilon)/2$ in the beginning, we have that

$$v(T,x;u_0,L)\geq e^{r\epsilon T}(\tilde{v}_\infty(T,x;u_0)-\epsilon)\geq e^{r\epsilon T}((1-\epsilon)/2-\epsilon)=e^{r\epsilon T}(1-3\epsilon)/2\geq e^{r\epsilon T}u_0(x),$$

for any $x\in\operatorname{supp}(u_0)$ and $T>T_0$. This together with Proposition 6.1 implies that for $L\gg1$, $\lambda(c,L)\geq r\epsilon>0$.

(5) By (1), $\lambda(c,L)$ is a principal eigenvalue. Hence there exists a $\phi>0$ such that

$$c\phi'(\xi)+\int_{\mathbb{R}}\kappa(\xi-\eta)\phi(\eta)d\eta-\phi(\xi)+f(\xi,0)\phi(\xi)=\lambda(c,L)\phi(\xi). \tag{6.28}$$

Let $\mu^*>0$ and c^* be as in (2.5), i.e., $c^*=\frac{\int_{\mathbb{R}}e^{-\mu^*z}k(z)dz-1+r}{\mu^*}$. Let (λ^*,ψ) be such that $\psi=e^{\mu^*\xi}$ and $\lambda^*=\int_{\mathbb{R}}e^{-\mu^*z}k(z)dz-1+r-\mu^*c$. Then (λ^*,ψ) satisfies that

$$-c\psi'(\xi)+\int_{\mathbb{R}}\kappa(\xi-\eta)\psi(\eta)d\eta-\psi(\xi)+r\psi(\xi)=\lambda^*\psi(\xi), \tag{6.29}$$

because

$$\begin{aligned}
&-c(e^{\mu^*\xi})'+\int_{\mathbb{R}}\kappa(\xi-\eta)e^{\mu^*\eta}d\eta-e^{\mu^*\xi}+re^{\mu^*\xi}\\
&=-\mu^*c(e^{\mu^*\xi})+\int_{\mathbb{R}}\kappa(\xi-\eta)e^{\mu^*\eta}d\eta-e^{\mu^*\xi}+re^{\mu^*\xi}\\
&=(-\mu^*c+\int_{\mathbb{R}}\kappa(\xi-\eta)e^{-\mu^*(\xi-\eta)}d\eta-1+r)e^{\mu^*\xi}\\
&=\lambda^*\psi(\xi).
\end{aligned}$$

Multiply (6.28) by ψ and (6.29) by ϕ, integrate both sides of the above equations and subtract, then we have that

$$\begin{aligned}
(\lambda(c,L)-\lambda^*)\int_{\mathbb{R}}\psi\phi d\xi=&\int_{\mathbb{R}}[c\phi'(\xi)+\int_{\mathbb{R}}\kappa(\xi-\eta)\phi(\eta)d\eta-\phi(\xi)+f(\xi,0)\phi(\xi)]\psi(\xi)d\xi\\
&-\int_{\mathbb{R}}[-c\psi'(\xi)+\int_{\mathbb{R}}\kappa(\xi-\eta)\psi(\eta)d\eta-\psi(\xi)+r\psi(\xi)]\phi(\xi)d\xi\\
=&\int_{\mathbb{R}}c[\phi'(\xi)\psi(\xi)+\psi'(\xi)\phi(\xi)]d\xi\\
&+\int_{\mathbb{R}}\kappa(\xi-\eta)\phi(\eta)\psi(\xi)d\eta d\xi-\int_{\mathbb{R}}\kappa(\xi-\eta)\phi(\xi)\psi(\eta)d\eta d\xi\\
&+\int_{\mathbb{R}}[f(\xi,0)-r]\phi(\xi)\psi(\xi)d\xi.
\end{aligned}$$

In addition, by the results of (1) in this theorem, we have that ϕ is bounded and

$$\limsup_{\xi\to\infty} \frac{\phi(\xi)}{e^{\mu_-(\lambda(c,L))\xi}} \le \tilde{M}_+.$$

Then $\lim_{\xi\to-\infty} \phi(\xi)\psi(\xi) = \lim_{\xi\to-\infty} \phi(\xi)e^{u^*\xi} = 0$ and with $\mu^* < -\mu_-(\lambda(c,L))$, we have that

$$0 \le \lim_{\xi\to\infty} \phi(\xi)\psi(\xi) = \lim_{\xi\to\infty} \phi(\xi)e^{u^*\xi} \le \lim_{\xi\to\infty} \tilde{M}_+ e^{(\mu^*+\mu_-(\lambda(c,L)))\xi} = 0.$$

Therefore we have that $\int_{\mathbb{R}} c[\phi'(\xi)\psi(\xi)+\psi'(\xi)\phi(\xi)]d\xi = \int_{\mathbb{R}} c[\phi(\xi)\psi(\xi)]'d\xi = 0$. By Fubini's theorem, $\int_{\mathbb{R}} \kappa(\xi-\eta)\phi(\eta)\psi(\xi)d\eta d\xi = \int_{\mathbb{R}} \kappa(\eta-\xi)\phi(\xi)\psi(\eta)d\eta d\xi$. With $k(z)=k(-z)$, we have that $\int_{\mathbb{R}} \kappa(\xi-\eta)\phi(\eta)\psi(\xi)d\eta d\xi - \int_{\mathbb{R}} \kappa(\xi-\eta)\phi(\xi)\psi(\eta)d\eta d\xi = 0$. Therefore we have that

$$(\lambda(c,L)-\lambda^*)\int_{\mathbb{R}} \psi\phi d\xi = \int_{\mathbb{R}} [f(\xi,0)-r]\phi(\xi)\psi(\xi)d\xi \le 0,$$

and thus $\lambda(c,L) \le \lambda^*$. Since $\lambda^* = \int_{\mathbb{R}} e^{-\mu^* z}k(z)dz - 1 + r - \mu^* c = \mu^*(c^*-c)$, $\lambda^* < 0$ if $c > c^*$. This implies that $\lambda(c,L) < 0$ if $c > c^*$. □

6.3. Applications of principal eigenvalue theory

In this subsection, we discuss the persistence and extinction in (1.5) by applying the principal eigenvalue theory established in the previous subsection. Our main results of this subsection are stated in the following Theorem 6.3. In the statement of Theorem 6.3, we use $\mu_\pm(\lambda(c,L))$ which were defined in (1.9). Let $v(t,\xi;v_0)$ be the solution of (1.5) with $v(0,\xi;v_0) = v_0(\xi) \in X^+$.

Theorem 6.3.

(1) (Persistence) If $\lambda(c,L) > 0$, then there is a positive stationary solution of (1.5), and for any $K > 0$ and $v_0 \in Int(X^+)$ satisfying $\liminf_{\xi\to\infty} \frac{v_0(\xi)}{e^{\mu_-(\lambda(c,L))\xi}} > 0$ and $\liminf_{\xi\to-\infty} \frac{v_0(\xi)}{e^{\mu_+(\lambda(c,L))\xi}} > 0$,

$$\liminf_{t\to\infty} \inf_{|\xi|\le K} v(t,\xi;v_0) > 0.$$

(2) (Extinction) If $\lambda(c,L) \le 0$, then for any $v_0 \in X^+$,

$$\lim_{t\to\infty} \sup_{\xi\in\mathbb{R}} v(t,\xi;v_0) = 0.$$

Proof. (1) First, assume $\lambda(c,L) > 0$. Let $\phi(\xi)$ be a corresponding positive eigenfunction with $\|\phi\|_\infty = 1$. Let $\underline{v}(\xi) = \alpha\phi(\xi)$ for $\alpha > 0$. Then we have that

$$\begin{aligned}
&-(c\underline{v}'(\xi) + \int_{\mathbb{R}} \kappa(\eta-\xi)\underline{v}(\eta)d\eta - \underline{v}(\xi) + f(\xi,\underline{v})\underline{v}(\xi))\\
&= -\alpha(c\phi'(\xi) + \int_{\mathbb{R}} \kappa(\eta-\xi)\phi(\eta)d\eta - \phi(\xi) + f(\xi,\alpha\phi)\phi(\xi))\\
&= -\alpha(c\phi'(\xi) + \int_{\mathbb{R}} \kappa(\eta-\xi)\phi(\eta)d\eta - \phi(\xi) + f(\xi,0)\phi(\xi)) + \alpha(f(\xi,0) - f(\xi,\alpha\phi))\phi(\xi)\\
&= -\alpha\phi(\xi)(\lambda(c,L) - (f(\xi,0) - f(\xi,\alpha\phi))).
\end{aligned}$$

Note that there is $\alpha_0 > 0$ such that $f(\xi,0) - f(\xi,\alpha\phi) < \lambda(c,L)$ for $0 < \alpha < \alpha_0$, and thus

$$-(c\underline{v}'(\xi) + \int_{\mathbb{R}} \kappa(\eta-\xi)\underline{v}(\eta)d\eta - \underline{v}(\xi) + f(\xi,\underline{v})\underline{v}(\xi)) \le 0.$$

This implies that $\underline{v}$ is a sub-solution of (1.5) for $0 < \alpha \le \alpha_0$.

Next, choose $M > \max\{1, \alpha_0\}$. Let $\bar{v}(\xi) \equiv M$. Note that $f(x, M) < 0$ if $M > 1$. Thus $-(c\bar{v}'(\xi) + \int_{\mathbb{R}} \kappa(\eta - \xi)\bar{v}(\eta)d\eta - \bar{v}(\xi) + f(\xi, \bar{v})\bar{v}(\xi)) = -f(\xi, M)M \geq 0$. Hence $\bar{v}(\xi)$ is a super-solution of (1.5). Note that $\bar{v} > \underline{v}$. Let $v(t, \xi; \underline{v})$ be the solution of (1.5) with $v(0, \xi; \underline{v}) = \underline{v}(\xi)$. Then by the comparison principle (Proposition 2.1)

$$\bar{v} \geq v(t_2, \xi; \underline{v}) = v(t_1, \xi; v(t_2 - t_1, \cdot; \underline{v})) \geq v(t_1, \xi; \underline{v}), \quad \forall\, 0 < t_1 < t_2,\ \xi \in \mathbb{R}$$

and

$$\bar{v} \geq v(t, \xi; \underline{v}) = v(t, \xi; \alpha\phi(\xi)) \geq \underline{v} = \alpha\phi(\xi) \quad \forall\, t > 0, \xi \in \mathbb{R},\ 0 < \alpha \leq \alpha_0.$$

It then follows that $\Phi(\xi)$ is a positive stationary solution of (1.5), where

$$\Phi(\xi) = \lim_{t\to\infty} v(t, \xi; \underline{v}).$$

Suppose that $v_0 \in Int(X^+)$ with $\liminf_{\xi\to\infty} \frac{v_0(\xi)}{e^{\mu_-(\lambda(c,L))\xi}} > 0$ and $\liminf_{\xi\to-\infty} \frac{v_0(\xi)}{e^{\mu_+(\lambda(c,L))\xi}} > 0$. Recall that there are $\tilde{M}_\pm$ such that

$$\limsup_{\xi\to\infty} \frac{\phi(\xi)}{e^{\mu_-(\lambda(c,L))\xi}} \leq \tilde{M}_+ \quad \text{and} \quad \limsup_{\xi\to-\infty} \frac{\phi(\xi)}{e^{\mu_+(\lambda(c,L))\xi}} \leq \tilde{M}_-.$$

Then there are $C > 0$ and $0 < \alpha_1 \leq \alpha_0$ such that for all $0 < \alpha \leq \alpha_1$ and $|\xi| > C$, $v_0(\xi) \geq \underline{v} = \alpha\phi(\xi)$. Let $\alpha_2 = \frac{\min_{\xi\in[-C,C]}\{v_0(\xi)\}}{\max_{\xi\in[-C,C]}\{\phi(\xi)\}}$. Then for $0 < \alpha < \alpha_2$, we also have that $v_0(\xi) \geq \alpha\phi(\xi)$, for $|\xi| \leq C$. Thus, for $0 < \alpha < \min\{\alpha_1, \alpha_2\}$, we have that

$$v_0(\xi) \geq \underline{v} = \alpha\phi(\xi), \quad \forall\, \xi \in \mathbb{R}.$$

It then follows by comparison principle (Proposition 2.1) that

$$v(t, \xi; v_0) \geq v(t, \xi; \underline{v}) \geq \alpha\phi(\xi), \quad \forall\, t > 0, \xi \in \mathbb{R}.$$

Thus (1) follows.

(2) We start the proof by making a stronger assumption that $\lambda(c, L) < 0$. By Proposition 6.1, $\|V(t; c, a)\| \leq e^{\lambda(c,L)t} \to 0$, as $t \to \infty$. Then for any $v_0 \in X^+$,

$$\lim_{t\to\infty} \|V(t; c, a)v_0\| = 0.$$

Note that for $t > 0$,

$$v(t, \xi; v_0) = V(t; c, a)v_0 + \int_0^t V(t - s; c, a)[f(\xi, v(s, \xi; v_0)) - f(\xi, 0)]v(s, \xi; v_0)ds.$$

Since $f_u(x, u) \leq 0$ in (H2) and $v(t, \xi; v_0) \geq 0$, this implies that

$$\int_0^t V(t - s; c, a)[f(\xi, v(s, \xi; v_0)) - f(\xi, 0)]v(s, \xi; v_0)ds \leq 0.$$

Therefore we have that

$$0 \leq v(t, \xi; v_0) \leq V(t; c, a)v_0, \quad \forall\, t \geq 0,\ \xi \in \mathbb{R}.$$

By the squeeze theorem, $\lim_{t\to\infty} v(t, \xi; v_0) = 0$.

Now we assume that $\lambda(c, L) = 0$. Let ϕ be a positive principal eigenfunction associated with 0, that is,

$$c\phi'(\xi) + \int_{\mathbb{R}} \kappa(\eta - \xi)\phi(\eta)d\eta - \phi(\xi) + f(\xi, 0)\phi(\xi) = 0.$$

Let $\psi(\xi) = \phi(-\xi)$ and then $\psi(\xi)$ satisfies that

$$-c\psi'(\xi) + \int_{\mathbb{R}} \kappa(\eta - \xi)\psi(\eta)d\eta - \psi(\xi) + f(\xi, 0)\psi(\xi) = 0. \tag{6.30}$$

Choose M large enough such that $\bar{v} = M$ is a super-solution of Eq. (1.5) and $0 \leq v_0 \leq \bar{v} = M$. Then by the comparison principle (Proposition 2.1), $0 \leq v(t, \xi; v_0) \leq v(t, \xi; \bar{v}) \leq M$. Then $\lim_{t\to\infty} v(t, \xi; \bar{v})$ exists and let $w(\xi) = \lim_{t\to\infty} v(t, \xi; \bar{v})$ that satisfies

$$cw'(\xi) + \int_{\mathbb{R}} \kappa(\eta - \xi)w(\eta)d\eta - w(\xi) + f(\xi, w)w(\xi) = 0. \tag{6.31}$$

By the similar arguments in the proof of item (3) of Theorem 6.2, multiply (6.31) by ψ and (6.30) by w, integrate both sides of the above equations and subtract, then we have that

$$\int_{\mathbb{R}} (f(\xi, w) - f(\xi, 0))w(\xi)\psi(\xi)d\xi = 0, \tag{6.32}$$

Note that $w(\xi) \geq 0$, $\psi > 0$ and by (H2), $f(\xi, w) \leq f(\xi, 0)$. From (6.32), if $w(\xi) > 0$, then we must have $f(\xi, w) = f(\xi, 0)$, which causes a contradiction. Therefore, we must have that $w(\xi) = 0$ and so $\limsup_{t\to\infty} v(t, \xi; v_0) \leq \lim_{t\to\infty} v(t, \xi; \bar{v}) = 0$. □

Remark 6.2. In the case that $\lambda(c, L) > 0$, it remains an open question whether Theorem 6.3(1) holds for any $v_0 \in X^+ \setminus \{0\}$.

Corollary 6.1. *$L^* = L^{**}$, and $L^* \to \infty$ as $c \to (c^*)^-$.*

Proof. First, we prove that $L^* = L^{**}$. For any $L > L^{**}$, $\lambda(c, L) > 0$. By Theorem 6.3, (1.5) has a positive stationary solution. Then by Theorem 5.1, we must have $L^{**} \geq L^*$.

Conversely, for any $L > L^*$, by Theorem 5.1, (1.5) has a positive stationary solution. Then by Theorem 6.3, we must have $L^* \geq L^{**}$. It then follows that $L^* = L^{**}$.

Next, we prove that $L^* \to \infty$ as $c \to (c^*)^-$. To indicate the dependence of L^* on c, we denote it by $L^*(c)$. Assume that there is $c_n \to (c^*)^-$ such that $L^*(c_n) \to \tilde{L}^* < \infty$ as $n \to \infty$. Fix $L > \tilde{L}^*$. Then, without loss of generality, we may assume that $\lambda(c_n, L) > 0$ for $n \geq 1$. By Theorem 6.2(2), we have that $\lambda(c^*, L) \geq 0$ for $L > \tilde{L}^*$. By Theorem 6.2(3), we must have that $\lambda(c^*, L) > 0$ for $L > \tilde{L}^*$. By Theorem 6.2(5), $\lambda(c, L) < 0$ for any $c > c^*$ and $L > 0$. Then by Theorem 6.2(2) again, $\lambda(c^*, L) \leq 0$ for $L \geq \tilde{L}^*$, which is a contradiction. □

Acknowledgments

This research is supported in part from NSF-DMS-1411853 of Patrick De Leenheer. The authors would like to thank the anonymous referees for their time carefully reading our manuscript, and for providing valuable suggestions which considerably improved the presentation of the paper.

References

[1] H. Berestycki, J. Coville, H.-H. Vo, Persistence criteria for populations with non-local dispersion, J. Math. Biol. 72 (7) (2016) 1693–1745.
[2] J. Coville, L. Dupaigne, Propagation speed of traveling fronts in non local reaction–diffusion equations, Nonlinear Anal. 60 (2005) 797–819.
[3] J. Coville, J. Dávila, S. Martínez, Pulsating fronts for nonlocal dispersion and KPP nonlinearity, Ann. Inst. H. Poincaré Anal. Non Linéaire 30 (2013) 179–223.

[4] W. Shen, A. Zhang, Spreading speeds for monostable equations with nonlocal dispersal in space periodic habitats, J. Differential Equations 249 (4) (2010) 747–795.
[5] W. Shen, A. Zhang, Traveling wave solutions of monostable equations with nonlocal dispersal in space periodic habitats, Commun. Appl. Nonlinear Anal. 19 (2012) 73–101.
[6] W. Shen, A. Zhang, Stationary solutions and spreading speeds of nonlocal monostable equations in space periodic habitats, Proc. Amer. Math. Soc. 140 (2012) 1681–1696.
[7] K. Schumacher, Traveling-front solutions for integro-differential equations I., J. Reine Angew. Math. 316 (1980) 54–70.
[8] H. Berestycki, O. Diekmann, C.J. Nagelkerke, P.A. Zegeling, Can a species keep pace with a shifting climate? Bull. Math. Biol. 71 (2) (2009) 399–429.
[9] M.A. Lewis, N.G. Marculis, Z. Shen, Integrodifference equations in the presence of climate change: persistence criterion, travelling waves and inside dynamics, J. Math. Biol. 77 (6–7) (2018) 1649–1687.
[10] B. Li, S. Bewick, J. Shang, W.F. Fagan, Persistence and spread of a species with a shifting habitat edge, SIAM J. Appl. Math. 5 (2014) 1397–1417.
[11] A.B. Potapov, M.A. Lewis, Climate and competition: the effect of moving range boundaries on habitat invasibility, Bull. Math. Biol. 66 (5) (2004) 975–1008.
[12] Y. Zhou, M. Kot, Discrete-time growth-dispersal models with shifting species ranges, Theoret. Ecol. 4 (2011) 13–25.
[13] C. Hu, B. Li, Spatial dynamics for lattice differential equations with a shifting habitat, J. Differential Equations 259 (2015) 1967–1989.
[14] W. Li, J. Wang, X. Zhao, Spatial dynamics of a nonlocal dispersal population model in a shifting environment, J. Nonlinear Sci. 28 (2018) 1189–1218.
[15] D. Henry, Geometric Theory of Semilinear Parabolic Equations, in: Lecture Notes in Math., vol. 840, Springer-Verlag, Berlin, 1981.
[16] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1983.
[17] L. Kong, W. Shen, Liouville type property and spreading speeds of KPP equations in periodic media with localized spatial inhomogeneity, J. Dynam. Differential Equations 26 (1) (2014) 181–215.
[18] J. Wang, X. Zhao, Uniqueness and global stability of forced waves in a shifting environment, Proc. Amer. Math. Soc. 147 (2019) 1467–1481.
[19] J. Coville, On a simple criterion for the existence of a principal eigenfunction of some nonlocal operators, J. Differential Equations 249 (11) (2010) 2921–2953.
[20] K. Deimling, Nonlinear Functional Analysis, Springer, 1985.
[21] R. Bürger, Perturbations of positive semigroups and applications to population genetics, Math. Z. 197 (1988) 259–272.
[22] V. Hutson, W. Shen, G.T. Vickers, Spectral theory for nonlocal dispersal with periodic or almost-periodic time dependence, Rocky Mountain J. Math. 38 (4) (2008) 1147–1175.