INSTRUCTOR HANDBOOK
to accompany
Discrete Mathematics Through Guided Discovery: Class notes for MTH 355 [*]

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1 Introduction

The notes are designed as the written course material for an introductory course in discrete mathematics for math majors who have little experience with mathematical proof. At Oregon State, the prerequisites are completion of the calculus sequence as well as a first course in applied differential equations and a first course in matrix algebra. The typical student is a mathematics major who is concurrently taking the first term of advanced calculus. The notes are an adaptation of the book *Combinatorics Through Guided Discovery* by Kenneth Bogart, for which the prototype was a smaller elective course with more mathematically sophisticated students.

The material is developed in problem sequences that are intended to develop expertise with using definitions and constructing mathematical proofs in the context of discrete mathematics. We have taught from these notes for several years, and have found them to foster and develop some important mathematical instincts in a different way from our other courses required for math majors. Among these instincts are: checking small cases first; formulating and testing conjectures; emphasizing the importance of testing for counterexamples when trying to construct a proof. By the end of the term, our students knew not to expect problems that were simply quick applications of cookbook algorithms and many voiced at least some satisfaction with this feature of the course.

1.1 How do these notes differ from the original?

Ken Bogart's website, `http://www.math.dartmouth.edu/∼kpbogart`, contains a copy of his book, which is comprised of six chapters and three appendices. His first two appendices consist of review material on relations and on mathematical induction, and the third appendix on exponential generating functions is intended as a supplement for graduate students.

At Dartmouth and most of Ken's other test sites, the notes were used in small elective courses. In contrast, the course at Oregon State is required of all math majors and we cannot assume the review material on relations and mathematical induction as a course prerequisite. Consequently, the review material in Ken's first two appendices has been incorporated as problem sequences in the main body of the notes and using this elementary material is a continuing theme in the adaptation. For instance, there are more questions testing the definition of function, and Ken's sections on distributions have been re-written from the perspective of equivalence relations.

Some of these changes reflect the fact that Oregon State's course is intended to be an introduction to discrete mathematics rather than a course in enumerative combinatorics. The table in Appendix A gives a comparison of the topics covered in Ken's notes with the topics our notes.

2 Overall advice

2.1 Advice on using the notes

Experience has shown that about 1.5 weeks should be devoted to each chapter covered. This rate allows enough time for group work, while requiring an amount of outside work commensurate with a third-year course. At this rate, not all chapters can be covered in a one-quarter course and also within each chapter not all problems can be worked.

It is important for students to proceed linearly through the first four chapters of the book (while returning to earlier problems that stumped them). This linear development is necessary not only because the material is interconnected, but also because problems in later chapters are sometimes stated more tersely or in other ways require a facility developed in earlier problem sequences.

There is some redundancy in the problem sequences of the first two chapters, enough that more experienced students can be steered to move quickly through some of this material and then encouraged to work on the optional sections. Students should understand the content of every problem marked with a solid bullet, even if they have not solved the problem.

2.2 Class time

In order to provide sufficient immersion in the problems, Oregon State's course meets twice a week in 75-minute sessions. (In the original classes met in 100-minute blocks, but it was difficult for students to remain on task for that amount of time.) Students spend class time working in four-person groups. These groups usually have been formed by the instructor and are changed about every week in the beginning of the term.

With our class meeting on Mondays and Wednesdays, the usual format has been to assign a range of problem numbers on Monday, a minimal assignment to be completed by the beginning of the next Monday's class. Occasionally the list of mandatory problems is modified on Wednesday, but in general it seems wise to set weekly expectations that students can and must strive to meet.

A small percentage of class time (not necessarily every day) has been devoted to whole-class discussion. This is the feature with the highest degree of variability among instructors—both how this time is used and also its frequency. For instance, an instructor might expect every student to present at least one problem to the class every term, while in other classes it has been used in a more ad-hoc way. The time can be used to model effective group work, especially in the beginning of the term.

2.3 Course grades

In our classes we used a fairly typical allocation of points on which the course grade was determined: 50% graded problems; 20% midterm; 30% comprehensive final. Although students work together in class and are encouraged to work together outside class, the final written draft of their problem solutions must be done independently. In some classes the problem grade included in-class presentations. We do not advocate giving a grade based exclusively on group work but it might be worthwhile to work some peer review into the grading scheme.

In Ken's method every problem received one the following grades: 0-5-9-10. In our classes, we graded only a subset of assigned problems and added a possible grade of 7. On every problem John graded he assigned two grades: one on mathematical content and the other specifically assessing writing.

Ken's method allowed unlimited re-submission of problems. We allowed a limited number of re-submissions within a specified amount of time. We have found that quick and frequent feedback is essential, and our goal was to collect written work on Monday and return the graded work at the next class meeting in two days.

2.4 Motivating students to do more

Students who understand the material more quickly can be encouraged to form groups that do more problems, for instance problems in the optional sections. There is general agreement that most students should be at about the same place most of the time—at a minimum, they should finish the chapters at about the same time. Working on the optional sections allows some students to do more problems without asynchrony.

The course format should allow for flexibility, but in practice it has been difficult to encourage students to do more. One successful strategy was to form groups with an eye to cultivating at least one group of students who want to solve non-routine problems. Students who are truly exceptional can be given a modified list of assigned problems, and possibly replace the regular final with take-home problems of greater difficulty.

3 An overview of content

Chapter 1: This chapter develops an understanding of the usefulness of partitions, especially as used in the Sum and Product Principles, and cultivates the use of functions in counting (especially the Bijection Principle).

Students need to be encouraged to look at small cases, especially when they're having trouble.

We suggest factorials and binomial coefficients not be used until they're developed in these notes. Some students have already taken an introductory probability course (MTH 361) and mistakenly think they already know the material in this chapter.

Chapter 2: The problems in this chapter involve working on inductive processes and the principle of mathematical induction. During the first term these notes were used, students objected that these problems were not of the proper form for induction problems. (Their earlier experience was with very formulaic problems.) In an attempt to address this, the notes have been revised to emphasize constructing inductive processes.

In addition to the more routine problems, the Product and Sum Principles are proved in this chapter. Students also conjecture and prove formulas for the number of subsets of a finite set and the number of functions between two finite sets. The chapter tries to develop an understanding of why counting the number of subsets is a special case of counting the number of functions, and that the General Product Principle is actually counting sets of functions.

Chapter 3: Continues with partitions by developing the use of equivalence relations in counting. For instance, they are used to count the number of k-subsets of an n-set as well as multisets and other examples of distributions. Binomial coefficient notation is established.

Chapter 4: The first two sections of this chapter on graph theory give more practice with mathematical induction, and the section on labelled trees works with the Bijection Principle. Sections 4.3, 4.5, 4.6 are among the most prescriptive in the book, and they can be used as a respite. The material on monochromatic subgraphs, minimal spanning tree, and shortest paths are independent of the rest of the notes and so can be omitted for the sake of time. However, many would argue that they are expected to be covered in any introductory course in discrete mathematics.

The remaining three chapters are less mainstream. We recommend the summary material in Section 7.1 be included.

Chapter 5: Using generating functions is an important counting technique, and the approach given here originated with Pólya in 1956. The algebra of formal power series is developed. Generating functions are used to obtain the closed-form solution of first-order recurrences and second-order homogeneous recurrences.

Chapter 6: The Principle of Inclusion-Exclusion is a generalization of the Sum Principle. A good application is counting the number of onto functions. This is considered in a different way in Chapter 7.

Chapter 7: The first section expects students to pull together all the information on distributions that has already been developed. The remainder of the chapter includes work on Stirling numbers (the number of n-part partitions of a k-set) and Bell numbers (the number of partitions of a k-set).

3.1 More specific comments from previous instructors

Chapter 1: Problem 9: Some students have trouble deciding whether the "order of the ordering" matters. This is an example of a place where the notes are intentionally terse. The real-world solution applies.

Problem 28 is recommended. There's a lot here for the students to sort through.

Student problems with the Bijection Principle in problems of the sort where students formulate a conjecture for the size of a set and then are asked to establish a bijection between that set and another set whose size they know. The following difficulties often occur: Students say there must be a bijection because the sets have the same size, or they write down any map between them and say it must be a bijection because the sets have the same size. The development of these problems has been revised often enough that we think the problem is inherent in the topic, and advise instructors to use whole-class discussion to emphasize the specific steps [Clear definition of the function; proof of 1-1; proof of onto].

Reviewers of the adaptation especially like Section 1.6, new to the adaptation.

Chapter 2: Students have trouble distinguishing between recognizing the inductive process involved and proving the inductive step. For instance, they try to try to do it in tandem. Problem 56 attempts to address this, and we recommend that problem.

In some problems the students thought identifying the process was a complete proof. For instance, that happened in an earlier version of Problem 59 on compositions of integers and so now we've been more explicit.

Problem 60: Many students don't realize that order is important, and often give an argument supporting the number 2^n.

Problem 66: (Proof of the Sum Principle by induction) This turned out to be a stretch for them. Some difficulties: The statement to be proved requires a universal quantifier; settling on a variable for induction; the "strong" principle is needed.

Problems like Problem 82 are easy for them, but very few students state they are using the General Product Principle in their arguments.

Chapter 3: Problem 95: Good practice with using notation and sorting through definitions. For instance, students make mistakes like beginning with $x \in C_x$.

For the sake of time, the section on Pascal's triangle may be omitted but it does contain the Binomial Theorem.

Some classes/groups especially enjoyed Problem 111.

Chapter 4: This chapter gives more practice with induction. Proofs of graph theory properties by induction have a different character. For instance, to construct a proof by induction on the number of edges you must begin with an arbitrary graph with n vertices and identify how the information from $n-1$ vertices is used. Many of the students began with an arbitrary graph with $n-1$ vertices and then talked about the "next" graph. Although this is now addressed explicitly, students continue to have difficulties.

Regarding the definition of walk: Since it's given as a sequence, there's an inherent direction but thinking in terms of direction is artificial in a graph whose edges are not directed.

Chapter 5: The section on pictures of trees in independent. Section 5.3 on solving recurrences is more prescriptive than others. (Some students in the original class commented how it pointed out how much trouble they had with simple algebraic manipulation...)

4 Chapter Summaries

Many of our students miss (and need) the instructor summarizing that is inherent in instructor-led courses. The following pages contain chapter review sheets. The TEXfiles are available upon request.

In the evaluation of the adaptation it was suggested that the students themselves do a review at the end of each chapter. For instance, at the beginning of the term they could be told that at the end of each chapter they would be expected to: create a dictionary of new terms; list the new theorems they have learned; list the new proof techniques in the chapter. The reviewer did not have copies of our review sheets, where this is exactly what we've done for the students. Since our students seem to need this direction (or, have become used to it), we're inclined to continue to supply these sheets.

REVIEW FOR CHAPTER 1

A short summary: In this chapter you developed some basic counting principles. In particular, uses of ordered pairs (The Product Principle) and set partitions (The Sum Principle) were developed and used. Also, the significance of functions in counting (including the Bijection Principle) was introduced.

New terminology: You should be able to define and use the following terms: function; one-to-one function; onto function; bijection; disjoint sets; partition of a set.

Basic Counting Principles:

The Product Principle If a finite set S is partitioned into m blocks of the same size n, then S has size mn.

The Sum Principle For any partition of a finite set S, the size of S is the sum of the sizes of the blocks of the partition.

The Bijection Principle Two sets have the same size if and only if there is a bijection between them.

The Pigeonhole Principle If a set with more than n elements is partitioned into n blocks, then at least one block has more than one element.

Major results you proved in this chapter:

1. Any set with n elements has 2^n subsets.

REVIEW FOR CHAPTER 2

A short summary: In this chapter, you reviewed the Principle of Mathematical Induction in the context of understanding underlying inductive processes. At first you used the so-called *simple* principle, but this was superseded by the *strong* principle that is just as easy to use and is often more convenient to apply. One of the important uses of mathematical induction in counting problems is that it can be used to prove the General Product Principle. Since the Product Principle from Chapter 1 is a special case of the general principle, from now on the term Product Principle will be reserved for this more general principle.

New terminology: You should be able to define and use the following terms: inductive process; first-order recurrence; permutation; k-permutation.

Basic Principles:

The Principle of Mathematical Induction To prove a sequence of statements indexed by integers $k \geq b$ it is sufficient to

1. State the sequence of statements to be proved;
2. (Base Step) Prove the statement is true for $k = b$;
3. (Inductive Step) Prove that (for any $N \geq b$) the truth of the statements for $k = b$, $k = b + 1$, ..., $k = N$ implies the statement with $k = N + 1$ is true;
4. (Inductive Conclusion) Conclude by the Principle of Mathematical Induction that every statement in the sequence is true.

The Product Principle Suppose you make a sequence of m choices, where the first choice can be made in k_1 ways, and for each way of making the first $i - 1$ choices, the i-th choice can be made in k_i ways, then the total number of different ways to make this sequence of m choices is $k_1 k_2 \cdots k_n$.

Major results you proved in this chapter:

1. The number of bijections on $[n]$ is $n!$. (Such bijections are also called permutations of the set $[n]$.)

2. The number of functions from $[m]$ to $[n]$ is n^m.

3. The number of k-element permutations of $[n]$ is $n(n-1)\cdots(n-k+1)$, which can be denoted as $n^{\underline{k}}$.

REVIEW FOR CHAPTER 3

A short summary: In this chapter you learned how the partition induced by an equivalence relation can be used in counting problems. In order to use this technique, you needed practice in identifying the underlying equivalence relation. By the end of the chapter you have proved the formula for calculating the number of k-subsets of $[n]$ (given by the binomial coefficient $\binom{n}{k}$) and the formula for counting multisets.

New terminology: You should be able to define and use the following terms: reflexive relation; symmetric relation; transitive relation; equivalence relation; equivalence class; binomial coefficient; ordered-functions; multisets.

Basic Principles:

Equivalence classes When R is an equivalence relation on the set $\mathcal{S}$, its set of equivalence classes forms a partition of $\mathcal{S}$.

Major results you proved in this chapter:

1. For any $k, n \geq 0$ with $k \leq n$, the number of k-element subsets of $[n]$ is

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}\,.$$

2. For any $k, n \geq 0$, there are $(k+n-1)^{\underline{k}}$ ordered-functions from $[k]$ to $[n]$.
3. For any $k, n \geq 0$, there are $\binom{n+k-1}{k}$ ways to choose a k-element multiset from $[n]$.

REVIEW FOR CHAPTER 4

A short summary: In this chapter you were introduced to the fundamentals of graph theory. Trees are among the most useful types of graphs, and minimal spanning trees in connected weighted graphs are used in many applications. You found that he Bijection Principle and the Principle of Mathematical Induction were crucial in the verification of important properties of graphs.

New terminology: You should be able to define and use the following terms: graph; degree of a vertex; connected graph; cycle; tree; spanning tree; weighted edge; minimal spanning tree.

Basic algorithm:

Dijkstra's Algorithm

Major results you proved in this chapter:

1. The sum of the degrees of the vertices in a graph equals twice the number of edges.
2. Any tree with n vertices has $n-1$ edges.
3. The number of labelled trees on n vertices is n^{n-2}.

REVIEW FOR CHAPTER 5

A short summary: Picture enumerators were used to solve counting problems related to multisets. They were also used to motivate generating functions—formal power series whose coefficients record the elements of sequence of numbers. Formal power series can be used to solve a variety of counting problems, including obtaining an explicit formula for certain recurrences, for example, Binet's Formula. In order to work with generating functions you must understand the algebra of formula power series.

New terminology: You should be able to define and use the following terms: generating polynomial; generating function.

Basic Principles:

Addition of Formal Power Series

$$\left(\sum_{i=0}^{\infty} a_i x^i\right) + \left(\sum_{j=0}^{\infty} b_j x^j\right) = \sum_{k=0}^{\infty} (a_k + b_k) x^k .$$

Multiplication of Formal Power Series

$$\left(\sum_{i=0}^{\infty} a_i x^i\right) \cdot \left(\sum_{j=0}^{\infty} b_j x^j\right) = \sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} a_i b_{k-i}\right) x^k .$$

Product Principle of Picture Enumerators

Product Principle for Generating Functions

Major results you proved in this chapter:

1.
$$(1-x)^{-n} = \sum_{i=0}^{\infty} \binom{n+i-1}{i} x^i .$$

2. Binet's Formula: If F_n is the n-th Fibonacci number then

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1} .$$

REVIEW FOR CHAPTER 6

A short summary: In this chapter you reviewed the terminology of sets in order to develop and understand the Principle of Inclusion and Exclusion. You used this principle to determine the number of derangements and the number of onto functions.

New terminology: You should be able to define and use the following terms: set terminology (union, intersection, complement); derangement; connected components of a graph.

Basic Principles:

The Principle of Inclusion and Exclusion The number of elements in $\bigcup_{i=1}^{n} A_i$ is

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{\substack{S\subseteq[n]\\ S\neq\emptyset}} (-1)^{|S|-1} \left|\bigcap_{i\in S} A_i\right|.$$

The number of elements in the complement of $\bigcup_{i=1}^{n} A_i$ in a (universal) set A is

$$\left|\overline{\bigcup_{i=1}^{n} A_i}\right| = |A| - \sum_{\substack{S\subseteq[n]\\ S\neq\emptyset}} (-1)^{|S|-1} \left|\bigcap_{i\in S} A_i\right|.$$

Major results you proved in this chapter:

1. The number of derangements of $[n]$ is

$$\sum_{s=0}^{n}(-1)^s \frac{n!}{s!}.$$

2. The number of onto functions from $[k]$ to $[n]$ is

$$\sum_{s=0}^{n}(-1)^s \binom{n}{s} (n-s)^k.$$

REVIEW FOR CHAPTER 7

k objects and conditions on how they are received	n recipients and the mathematical model for distribution	
	Distinct	Identical
Distinct no conditions	n^k functions	NOT DONE (set partitions into $\leq n$ parts)
Distinct Each gets at most one	$n^{\underline{k}}$ k-element permutations	1 if $k \leq n$; 0 otherwise
Distinct Each gets at least one	$\sum_{s=0}^{n}(-1)^s\binom{n}{s}(n-s)^k$ onto functions	NOT DONE (set partitions into n parts)
Distinct Each gets exactly one	$k! = n!$ permutations	1 if $k = n$; 0 otherwise
Distinct, order matters	$(k+n-1)^{\underline{k}}$ ordered-functions	$\sum_{i=1}^{n} L(k,i)$ broken permutations ($\leq n$ parts)
Distinct, order matters Each gets at least one	$(k)^{\underline{n}}(k-1)^{\underline{k-n}}$ ordered-onto-functions	$L(k,n) = \binom{k}{n}(k-1)^{\underline{k-n}}$ broken permutations (n parts)
Identical no conditions	$\binom{n+k-1}{k}$ multisets	NOT DONE (number of partitions into $\leq n$ parts)
Identical Each gets at most one	$\binom{n}{k}$ subsets	1 if $k \leq n$; 0 otherwise
Identical Each gets at least one	$\binom{k-1}{n-1}$ compositions (n parts)	NOT DONE (number of partitions into n parts)
Identical Each gets exactly one	1 if $k = n$; 0 otherwise	1 if $k = n$; 0 otherwise

A A tabular list of topics in the current version

Below is a list of topics according to Ken's sections with the corresponding sections in the current notes. Section numbers in *italics* are now optional.

Topic	Ken's	Adaptation
Sum Principle	1.2.1	1.2
Product Principle	1.2.1	1.2
Functions	A.1;1.2.2	A; 1.2, 1.3, 2.3.1
Directed graphs	A.1;1.2.2	A; 1.3
Bijection Principle	1.2.3	1.3
Counting subsets	1.2.4	3.3
Pascal's Triangle	1.2.5	3.3.1
Quotient Principle	1.2.6	(de-emphasized)
Lattice paths	1.3.1	*3.3.2*
Catalan numbers	1.3.1; 4.3.5	*3.3.2*
Binomial Theorem	1.3.2; 2.1.2	*3.3*
Pigeonhole Principle	1.3.3	1.4
Ramsey numbers	1.3.4 ;2.1.5	*1.5,3.5,4.4*
Mathematical Induction	B.1-B.2 ; 2.1.1	B; Chapter 2
Binomial coefficients	2.1.2	3.3
Inductive definition	2.1.3	Chapter 2
General Product Princ	*2.1.4*	2.3
Double Induction	2.1.5	*3.5*
Asymptotic combinatorics	2.1.6	*4.7*
Equivalence relations	A.2	Chapter 3
Recurrences	2.2; 4.3	2.2.1;5.3
Undirected graphs	2.3.1	4.1
Walks and paths	2.3.2	4.2
Labelled trees	2.3.3	4.3
Spanning trees	2.3.4	4.4
Minimal spanning trees	2.3.5	4.4
Deletion/contraction	2.3.6;5.3	4.4.1;*6.5.1*
Shortest paths	2.3.7	4.5

Topic	Ken's	Adaptation
Ordered-functions	3.1.2	3.4
Multisets	3.1.3; 5.2.1	3.4; 5.1
Compositions of integers	3.1.4	2.2; 3.3
Broken permutations	3.1.5	7.1
Stirling numbers	3.2	7.2
Partitions of integers	3.3	—
Picture functions	4.1.2	5.1
Generating polynomials	—	5.2.1
Generating functions	4.1.3	5.2.2
Formal power Series	4.1.4	5.2.2
Product Principle for gf	4.1.5	5.2.2; *5.2.3*
Extended Binomial Thm	4.1.6	—
Gf for integer partitions	4.2	—
Gf and recurrences	4.3	5.3
Partial fractions	4.3.4	5.3
Size of unions	5.1.1-5.1.2	1.2;6.1
Inclusion and Exclusion	5.1.3	6.2
Ménage Problem	5.2.2	6.4
Counting onto functions	5.2.3	6.3;7.2
Chromatic polynomial	5.2.4	6.5
Groups acting on sets	6.1-6.3	—

B More hints/suggestions

Notes from the summer workshop given in August 2003 by Ken Bogart and Karen Collins. More discussion can be found in the evaluator's report on Ken's website:
`http://www.math.dartmouth.edu/~kpbogart`

Grading The basics of problem grading in Ken's method.

- Ken's students were to do 20 problems per week, based on a 4-hours-a-week quarter course. This meant a total of 200 problems a term, which was lowered in practice to 190 problems.
- Ken graded each problem based on 0, 5, 9, 10 points. Here, 0 indicated essentially nothing was understood, as contrasted with 5 points earned by a solution which seems to be about ready to make a breakthrough. On the other hand, 10 was an essentially perfect paper, while papers graded as 9 had a few simple wording errors.
- Ken allowed unlimited resubmission of problems for a change of grade.

Promoting well-functioning groups The students were told:

- To make sure everyone understands what the sequence of problems is asking.
- To work together to get a believable solution to as many problems as possible during class.
- To work together to understand instructor feedback in grading.
- To work together on problems which were not graded until every group member is satisfied.

Large-group discussions For occasional whole-class discussion:

- The big picture would be the basis for discussion, and not just reviewing old problems.
- The class should be allowed to make some executive decisions. That is, there are logistical-type things that come up and most of them can be decided by the whole class.

Guiding for discovery

- Keep up the pace, with students always having slightly more than they're able to do.
- Be sure groups work in a way that is conducive to risk-taking.
- Students are told to do the problems in as many different ways as possible. In fact, more credit given for an extra solution than for original. (Adapted from Schoenfeld.)
- Occasionally switch groups so people get to work with new people.
- Ask the groups why they were doing something and how it helps the whole (idea from Schoenfeld) in order to give them control over their own thought processes.