Guided Group Discovery in a Discrete Mathematics Course for Mathematics Majors

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Dedicated to the memory of Kenneth P. Bogart

Abstract

This article discusses the use of Ken Bogart's method of guided group discovery in Oregon State's discrete mathematics course for math majors. We summarize the Bogart method and discuss some accommodations which we have made for our classes.

1 Introduction

In this article we discuss the use of guided group discovery in Oregon State University's discrete mathematics course for math majors. Since Fall 2003 this course has been taught at Oregon State using an ongoing modification of Kenneth P. Bogart's successful group discovery method and notes [2], "Teaching Introductory Combinatorics by Guided Group Discovery". Section 2 summarizes Ken's notes and method, and Sections 3 and 4 respectively contain the adaptation of his notes and the implementation of his method at Oregon State.

Ken's prototype was a small elective course in which the average entering student was very motivated to learn the material. In our department the course is required, and it is financially unrealistic for us to expect either very small classes or in-class assistance from a senior student. We think our modification is sufficiently general that it can be successfully used by other mathematics departments with similar student demographics. Since the adaptation is an ongoing project, the interested reader is referred to [6] for current information on its status.

Ken generously served as a consultant for our adaptation of his method, both informally in Fall 2003 and during the first year of the grant. We are grateful for his help.

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2 A Short Overview of the Bogart Course

The goal of Ken's project was to design notes and a method for teaching enumerative combinatorics in which "a large majority of the students would learn a large majority of the material of beginning combinatorics". The method was informed by earlier work in mathematics education, including Davidson's work on small group discovery [3]; Schoenfeld's research on problem solving [8]; and the work of Dubinsky and his collaborators on the genetic decomposition of mathematical knowledge [1, 4, 5].

The set of class notes is the only written resource for the course, and is comprised of inter-connected problem sequences in combinatorial mathematics and graph theory. Through working the problems students learn combinatorial processes and are guided to the discovery of general principles. Students are expected to complete 90% of the problems, with the proviso that they should not systematically avoid the more challenging problems. Most of the problems are preceded by symbols, whose meaning is summarized in the following table:

$\cdot$	essential for this or the next section
$\bullet$	essential
$\circ$	motivational material
$+$	summary
$\rightarrow$	especially interesting
$*$	difficult

This is an important feature of the notes, since the symbols assist the student and they also provide important cues for an instructor who wants to design a cohesive set of mandatory problems to be completed by the class.

The learning strategy of group discovery is based on the premise that most students learn well from working co-operatively with peers. Unlike many classes in which an expository lecture is the primary medium for conveying information, students in group discovery classes are actively involved in their own learning during class as well as outside the classroom. In the Bogart implementation of group discovery, most of the course grade comes from non-group work, including traditional in-class exams and the expectation that final solutions will be completed without consultation or collaboration.

During class time, the instructor must work to achieve a balance between giving students sufficient time to work successfully in groups and ensuring students are satisfactorily progressing through the material. Also, the notes are constructed so that an instructor can decide to guide some groups of students to push ahead and work on more difficult problems whose solution might even replace some required problems.

In the Bogart method, the instructor also guides by providing selective and thoughtful use of discussion which involves the whole class or a large portion of the class. This can range from a chance for students to discuss their understanding of a specific problem at the board to a review facilitated by the instructor. An interesting use of large-group discussion is to model successful group dynamics with no all-knowing participant. This can be especially effective when it is used at times when most of the students think they have hit a dead end on a problem sequence. As students are encouraged to explain what they and their group have tried or where they have trouble with a problem, the class can see that careful and patient listening to others can lead to progress.

Ken hoped guided discovery would strengthen the students' sense of their own responsibility for their learning, and that the instructor would foster an atmosphere where students themselves initiate discussions which involve a large percentage of the class. Careful use of large-group discussion is an integral part of the Bogart method, and can be an exemplary teaching method for the many future mathematics teachers who populate our classes.

3 The Adapted Notes

While using Ken's notes in Fall 2003, I was impressed by the student response to many of the problem sequences as well as by the amount of personal responsibility shown by most of the students. Some of this can be attributed to the type of problems, which are especially conducive to group discussion. This is true both because many of the problems are easily understood by students at different levels of mathematical maturity and because a slight change in wording can result in a major change in technique.

But we think most of the enthusiasm comes from the way the notes are constructed as well as the Bogart implementation of group discovery. As a check on our faithfulness to the original spirit, we asked Rosa Orellana to review our implementation from this point of view. Her report can be found at [6]. Rosa is a member of Dartmouth's faculty who has taught from Ken's notes and was also involved in their development.

3.1 Why adapt?

The prerequisites for Ken's course are: "comfort with sets, functions, and algebraic notation (including some summation notation); some experience with reading (and perhaps doing) proofs; and, ideally, a modest exposure to mathematical induction." A typical OSU math major takes this course concurrently with advanced calculus in the first term of their third year of study. As preparation, students have completed the calculus sequence and one course in matrix algebra. Since these are also service courses for engineering students, they provide little experience with mathematical proof and in general our students do not satisfy Ken's prerequisites when they begin this course.

Ken thought our class was the first in which his notes were used in a required course in which the usual enrollment[1] was at least twice as large as the maximum recommended by his original project. Ken encouraged us to apply for an NSF

[1] The usual class size has been in the mid- to high-20s, and in Fall 2007 there are two sections with about 16 students each.

grant to adapt his notes and method for the larger classes which are more typical of required courses at state universities.

There are other differences between the course (and classes) at Oregon State and the assumptions made by Ken and his advisory board on the intended use of his notes and method. In addition to the smaller classes and the different level of mathematical preparation, Ken's prototype class was an elective class and the average student began the course with more interest in the material. Our more extensive syllabus of discrete mathematics also required some adjustment.

Even with these differences, the basic philosophy of the notes transfers well and also complements the more traditional course in advanced calculus which is usually taken concurrently by our majors.

3.2 Changes in the structure of the notes

Before giving an overview of some of the differences in mathematical content, we first describe some changes we have made in the basic structure of the notes.

Ken's original notes assumed a familiarity with basic topics, such as relations and mathematical induction, that we cannot expect from our students when they enter this course. Although Ken's appendices contained some review of this material, flipping back and forth from those sections to the main book interfered with the students' appreciation of what they were doing and how things fit together. In the adaptation, this basic material has been expanded and incorporated into the fabric of the early chapters. Three review appendices remain, but we find an average student doesn't need to do more than scan them outside of class.

Because the topic of Ken's course was enumerative combinatorics, some of his problem sequences cover material that is not basic to our more general syllabus of discrete mathematics. In our adaptation, much of that advanced material has been moved to optional sections which are not required in the mainstream problem sequences.

Ken's first chapter was very long, and we have spread that material over three chapters in the adaptation. The new presentation encourages students to review frequently and tries to facilitate that summarizing process by packaging the material in more obvious units. In addition, since many of our students don't have sufficient mathematical maturity to ferret out the definitions and theorems from the exercises, we have added more formal definitions, more statements of theorems, and an occasional example of a complete mathematical proof. Ken's advisory board and evaluator advocated the inclusion of more summarizing material, and we have designed separate summarizing handouts for each chapter. They are available as TEXdocuments as well as in PDF format so that instructors can tailor the summaries to their individual courses.

Since this course is required for math majors, we've found that some problems had to be re-worded to distinguish between whether a complete proof is expected or whether a less-formal explanation would suffice. Developing and improving this skill is an important goal for the course, and we cannot assume our students enter with much proof experience.

3.3 Changes in content

In this section we discuss three specific areas in which changes were made in the notes for our classes: functions, equivalence relations, and mathematical induction.

As the students progressed through Ken's notes, we found serious deficiencies in their concept of function. A basic technique in enumerative combinatorics is to count the size of a finite set by establishing a bijection between it and another set whose size has already been established.[2] In order to successfully apply this technique, the problem solver's understanding of function must have evolved from plugging values into a given function through determining whether or not a relation is a function. In fact, he or she should recognize how the notion of function is useful for the problem and then construct the proper function; that is, the student must have at least arrived at the object stage of conceptualization [1]. In the original project Ken and his advisory board assumed that students would enter with sufficient knowledge to understand something as being characterized as a function without having a specific expression. However, from our listening to group discussions and our reading and commenting on their written work, it soon became clear that many of our students were far from this stage. Later course instructors and Professor Orellana have analyzed the adaptation from the point of view of addressing this problem while maintaining the interest of students whose concept of function has evolved into the process stage.

Another significant difference from the original notes is the adaptation's emphasis on equivalence relations. Many of Ken's problem sequences on the Quotient Principle and distributions have been changed to questions on equivalence relations. In addition to having wider application to equivalence classes of unequal size, using equivalence relations is an important skill to be obtained from a beginning course in discrete mathematics and is expected in the later required course in abstract algebra. The current edition has a full chapter on equivalence relations, absorbing much of the material from the original appendix as well as expanding problem sequences using the Quotient Principle. Ken supported this as a necessary and important change for a course in discrete mathematics.

Ken's notes introduce students to the Principle of Mathematical Induction through problem sequences which involve proof by contradiction. Specifically, problems ask students to prove a proposition indexed over all positive integers by contradicting the existence of a smallest counterexample. Once students have done problems using this technique, the commentary then introduces induction as the framework for a more natural direct proof. This was a successful new perspective for some advanced students in our classes, but others protested, saying that "this induction" looked nothing like the one they used in advanced calculus. These students were associating induction with formulaic exercises (such as finding a closed form for the sum of the first n positive integers) rather than as a method for proving a sequence of statements indexed by the positive integers.

[2]In the appendix to this article we give our modification of one of Ken's problem sequences using the Bijection Principle.

The problems in the notes were more varied than those they had encountered elsewhere, and we found that our students need more experience with problems that ask them to identify the underlying inductive process. Induction now occurs earlier in the adaptation, and more problems have been added, including a new sequence which builds student understanding of inductive processes. Ken and I were both intrigued by this from a pedagogical point of view as well as the more practical point of view of how it might be addressed.

4 The Adapted Method

The ongoing adaptation has been used in our discrete mathematics course since Fall 2004. The course is offered once a year, and three different instructors have used the adapted notes. In the longterm it is hoped that an instructor will teach the course for two consecutive years and then serve as a mentor for the next instructor (which could be in their second year if two sections are offered). The classes now meet twice a week in 75-minute sessions in order to provide sufficient immersion in the problems. For the first few years the classes met in 100-minute blocks, but we found it was more difficult for everyone to remain on task for that amount of time.

With our classes meeting on Mondays and Wednesdays, the usual format has been to assign a range of problem numbers on Monday, and this forms a minimal assignment to be completed by the beginning of the following Monday's class. Students spend class time working in four-person groups that have usually been formed by the instructor and whose composition changes about every other week.

Ken required students to solve 90% of the problems by the end of the term. In our ten-week courses, our classes regularly cover at most five of the seven chapters in the adaptation. It has been reasonable to expect the students to work about twenty problems a week for the first few weeks. As the term proceeds, problems become progressively harder, solutions often require making connections with earlier information, and completing about 12-15 problems is a reasonable expectation. Occasionally the list of mandatory problems is modified at the end of Wednesday's class, but in general it seems wise to set weekly expectations which the students strive to meet. Students are encouraged to work together outside class, but the final written draft of their problem solutions is expected to be done independently.

We have all found that quick and frequent feedback is essential, and our goal has been to collect written work on Monday and to return the graded work two days later in Wednesday's class. Ken's method allows for unlimited re-submission of problems, but in our classes both the number of re-submissions and the re-submittal window are restricted and varies according to the instructor.[3] All instructors have used our modification of Ken's 0-5-9/10 "triage"

[3] I allowed for a pre-specified number (about 8-10) of resubmits during the the ten-week quarter and they must be handed in within two weeks of my grading. A problem could be submitted more than twice.

grading-scale, adding a possible grade of 7 as a compensation for fewer resubmits. Some instructors have assigned a separate grade which specifically assesses student writing.

Which problems are graded? There is a great deal of variability in this. Some instructors collect all assigned problems and grade a subset, while others give a short list of problems at the end of Wednesday's class. When I've taught the course, I've purposely changed my expectations as the term progressed. For the first five weeks of the term, students turned in all assigned problems and I graded about five problems per week. Toward the end of the term, the students handed only in the smaller subset of problems which would be graded since by this time they should have learned what was expected for a complete mathematical solution. Although I encouraged the students to discuss their graded solutions with other members of their group before working on resubmits, time pressure from other courses made this an unrealistic expectation. The benefits of explaining work to peers and of analyzing others' work are many, enough that it might be worthwhile to encourage more of this type of collaboration. For instance, perhaps some peer review can be part of the grading scheme.

What about motivating students to do more? Students who understand the material more quickly can be encouraged to form groups which would do more problems, for instance ones in the optional sections. These sections contain problem sequences on material that is somewhat ancillary to our course (for instance Ramsey numbers) but can provide an intellectual challenge for these students. The second time I taught the course, three students formed a group which regularly worked on these sections. Two of those students were more experienced, and although sometimes the group work became close to a tutorial for the other student, all three students seemed satisfied with the arrangement. This group formed fairly early in the term and the midterm served as a check that they were learning the basic material. After the middle of the term we agreed that they would invest more time in harder problems with the understanding that they would be able to replace the final with take-home problems of greater difficulty. (And the less experienced student agreed to take the regular final.)

Ken and all of our instructors have agreed that in a larger classes (especially those with fluid group membership) it is important for most of the students to be at about the same place most of the time: at a minimum, they should finish the chapters at about the same time. Students proceeded linearly through the chapters, and were regularly encouraged to return to problems which at first stumped them.

Although it is always a small percentage of class time, the amount and frequency of whole-class discussion is the feature with the most variability in the different classes. For instance, some instructors expect every student to present at least one problem to the class every term.

There are possible advantages to a larger class. For example, I found there were always students who were eager to report on their group's work or to voice their group's unresolved questions in an end-of-class wrap up. For harder problems, several different approaches would sometimes result in spirited arguments that continued after class, and the larger class size might have increased the chance of that happening. Because there were always five or six groups, each group knew that class could not be a private group tutorial with the instructor and many learned to make good use of the group time as well as any whole-class discussion time. In addition, most students seemed not at all fazed (and obliged) when asked to switch groups, and this might have been more difficult if there were fewer people in the room.

5 Concluding Remarks

In this project Ken Bogart's notes and method were adapted for larger class size, weaker mathematical background and motivation, and some difference in course material. A continuing challenge is maintaining discovery while expanding the interstitial discussion and encouragement, each of which seems to be necessary in a class with five or more groups.

These notes foster and develop important mathematical instincts in different ways from our other courses for math majors. Among these are: checking small cases first; formulating and testing conjectures; the importance of testing for counterexamples in combination with trying to construct a proof. By the end of each term, all students knew not to expect problems which were simply quick applications of cookbook algorithms and many had voiced at least some satisfaction with this feature of the course.

The current adaptation and supporting materials are available on the website [6]. More information for instructors will be added to the site in coming years. Please contact me if you'd like more information.

References

[1] Mark Asiala, David J. DeVries, Ed Dubinsky, David Mathews, and Karen Thomas, *A framework for research and curriculum development in undergraduate mathematics education*, CBMS Issues in Mathematics Education **6** (1996), 1–32.

[2] Kenneth P. Bogart, *Teaching Introductory Combinatorics by Guided Group Discovery*, `www.math.dartmouth.edu/~kpbogart/`.

[3] Neil Davidson, *The small-group discovery method in secondary- and college-level mathematics*, Cooperative Learning in Mathematics, Wesley Publishing Company, 1990.

[4] Ed Dubinsky, *Teaching mathematical induction, I*, J. of Mathematical Behavior (1986), 305–317.

[5] ———, *Teaching mathematical induction, II*, J. of Mathematical Behavior (1989), 285–304.

[6] Mary E. Flahive, `www.oregonstate.edu/~flahivem/DMnotes.htm`.

[7] N. L. Hagelgans, *A practical guide to cooperative learning in collegiate mathematics*, The Mathematical Association of America, MAA Notes Number 37, Washington, DC, 1995.

[8] Alan Schoenfeld, *Mathematical Problem Solving*, Academic Press, 1985.

Appendix: An adapted problem sequence

The following is an example our adaptation of a problem sequence which builds on the notion of function and was designed from an actual class experience.

At the start of a two-hour meeting of the Fall 2003 class, most of the groups were beginning Ken's problem sequence on the enumeration of labeled trees (Problems 104 to 116.) By the end of the first hour, many groups saw a pattern emerging from counting the number for small vertex sets. Some students were ready to accept the conjecture without proof, while others wanted to find a general argument. After a short break, most of the students agreed to forge ahead and work on understanding Prüfer codes. A game evolved in which one group member would give a Prüfer code for their personal private tree and the rest of the group would then find the tree. By the end of the second hour, all of the groups had played the game and most students were convinced there would always exist a unique tree for every Prüfer code. Many of them left class trying to establish the necessary bijection by describing the sequences which were guaranteed to be Prüfer codes. Most of the students had progressed from barely knowing the definition of tree through making conjectures in the middle of class to leaving the class with some idea of how a proof might be constructed. The adapted sequence we give below uses this game. (The problem sequence refers to the following figure from an earlier section.)

Labeled Trees and Prüfer Codes

Next you will explore the idea of *labelled* trees. Figure 2 gives all different labellings of a fixed tree with 3 vertices. Notice that the convention for labelling the vertices of trees is that the tree which has edges between vertices 1 and 2 and between vertices 2 and 3 is different from the tree that has edges between vertices 1 and 3 and between vertices 2 and 3.

1. How many labelled trees are there on the vertex set $\{1, 2\}$? On the vertex set $\{1, 2, 3\}$? How many labelled trees are there on four vertices? How many labelled trees are there with five vertices? You don't have a lot of data to formulate a guess, but try to guess a formula for the number of labelled trees with vertex set $\{1, 2, \ldots, n\}$. When you get to four and especially five vertices, draw all the unlabelled trees you can think of, and then figure out in how many different ways you can put labels on the vertices.

Figure 1: Three different graphs

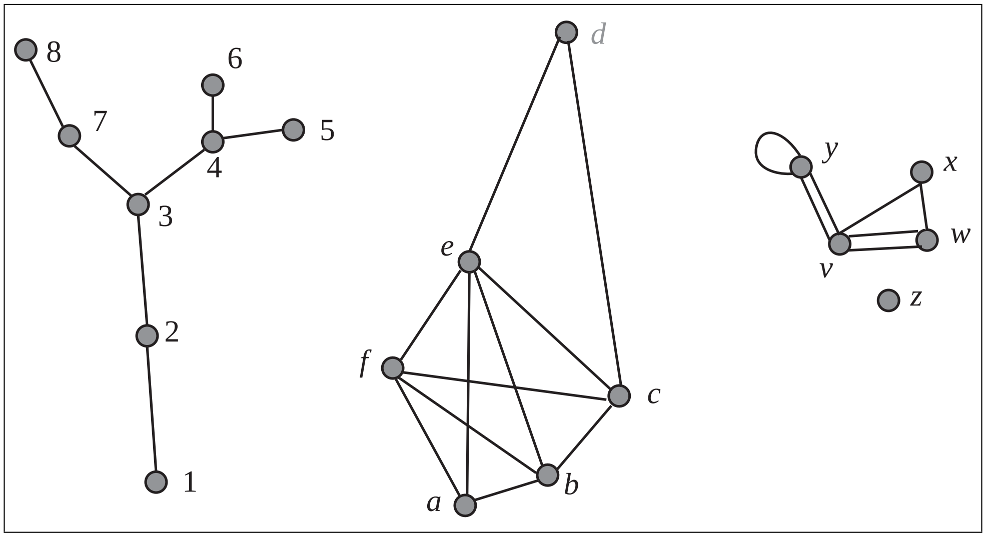
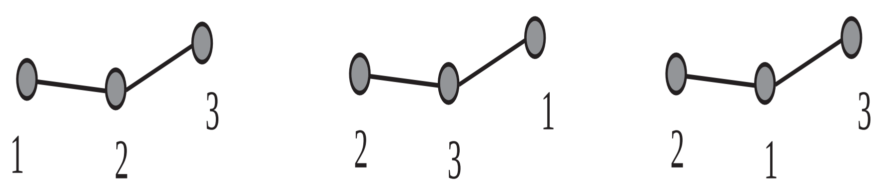


Figure 2: The three labelled trees on three vertices



The next problems will develop a method for proving the formula you just guessed in the last problem. In order to do this, an auxiliary sequence is defined.

Given a tree with $n \geq 2$ vertices which has been labelled in any way using the elements of $[n]$, define the auxiliary sequence $b_1, b_2, \ldots$ in the following inductive manner:

Step 1: If the tree has two vertices, the sequence consists of one term, the larger label, which means the sequence is $b_1 = 2$. Otherwise, let a_1 be the lowest-numbered vertex of degree 1 in the tree. (How do you know there is such a vertex?) Let b_1 be the label of the unique vertex in the tree adjacent to a_1 and write down b_1. (Why is b_1 unique?) For example, in the first graph in Figure 1, a_1 is 1 and b_1 is 2.

Step 2: Suppose a_1 through a_{i-1} have already been identified, and let a_i be the lowest-numbered vertex of degree 1 in the tree you get by deleting vertices a_1 through a_{i-1}. (How do you know the resulting graph is always a tree?) Let b_i be the unique vertex in this new tree adjacent to a_i. For example, in the first graph in Figure 1, $a_2 = 2$ and $b_2 = 3$. Then $a_3 = 5$ and $b_3 = 4$.

2. We use the letter B to stand for the sequence of b_is inductively obtained in this way. Use your earlier work to answer the questions posed in the above two-step algorithm.

3. For the tree (the first graph) in Figure 1, the sequence B is 2344378. At this point, work with your group to draw some other labelled trees on eight vertices and construct the sequence B associated with each tree.

4. How long is the sequence B computed from a labelled tree with n vertices?

5. From your examples, decide if you can predict the last member of the sequence B. Explain.

6. Is it possible for a_1 to be in B? Can you tell from B what a_1 is?

For a labelled tree T, the associated sequence $P(T) := b_1, b_2, \ldots, b_{n-2}$ is called a **Prüfer coding** or **Prüfer code** of T. For instance, the Prüfer code for the labelled tree T of Figure 1 is $P(T) = 234437$. Notice that we do not include the last term of B in the Prüfer code because we know it is n.

Let $\mathcal{S}$ be the set of all labelled trees on nine vertices. For each tree $T \in \mathcal{S}$, define $P(T)$ to be the Prüfer code for T.

7. Why is the relation $\{ (T, P(T)) \ : \ T \in \mathcal{S}\}$ a function with domain $\mathcal{S}$? Find a co-domain for this function. (At this point you're not asked to find the *smallest* co-domain.)

8. Play the following game in your group: In turn, each of you should secretly write down a tree, determine its Prüfer code, and then share the code with the whole group. The other members of the group then should find all labelled trees that have your sequence as its Prüfer code. How many labelled trees are found? What does your answer say about the function P?

9. Now, as a group write down any sequence of seven integers from $\{1, 2, \ldots, 9\}$. Try to find a tree $T \in \mathcal{S}$ for which your sequence is $P(T)$. Do this for several different sequences. Use this information to find the smallest co-domain for the function P?

10. Find a bijection between the set of labelled trees with n vertices and another set that you already now how to count.

11. Find the number of labelled trees with n vertices. Is this the formula you conjectured earlier in Problem 1?

The idea of writing the last sequence of problems as a game originated with the Fall 2003 Math 399 class at Oregon State.

In addition to providing a way to count labelled trees, there is a good bit of other interesting information encoded in the Prüfer code for a tree. You can begin to see this by working the next two problems and the problems in the optional section that follows them.

12. What can you say about the vertices of degree one from the Prüfer code for a tree labelled with the integers from 1 to n; that is, what vertex or vertices in the sequence $b_1, b_2, \ldots b_{n-1}$ can have degree 1?

13. What can you say about the Prüfer code for a tree in which exactly two vertices have degree 1? Does this characterize such trees?