

Some Observations on Teaching Induction

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We have taught mathematical induction in various courses over many years. The observations given here stem from an analysis of our experience, and especially from listening to math majors in recent discrete mathematics classes that we have recently taught using guided discovery.

Mathematical induction is often illustrated by an analogy to long rows of falling dominoes or to steps on infinite ladders. After listening to students, as they worked in groups and reading their subsequent written work, we've identified some subtleties of the technique that we now are more careful to emphasize with our classes.

Most students easily learn to apply the induction template to prove formulaic results. Indeed, Gauss' formula for the sum of the first n positive integers is a typical first problem in many textbooks. In that example, after verifying the base case, a student or a textbook might continue the induction argument as follows:

Assume inductively that:

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2} \quad (*)$$

Then

$$1 + 2 + \cdots + n + (n+1) = \frac{n(n+1)}{2} + (n+1) = \frac{(n+1)(n+2)}{2}$$

With these two sentences the induction step is advanced and the proof is completed. When we explicitly asked our students why these steps constitute a proof, most students were unable to explain. In fact, as we probed further most of them displayed a marked uncertainty with the correctness of their proofs. One reason for their ambivalence is the use of the word "assume" in the inductive hypothesis: many students find it difficult to distinguish between assuming something is true and knowing that it is true. This distinction requires a precision of language and thought that presses their mathematical experience and provides an opportunity for growth.

This example also illustrates one of the problems with restricting the first examples on induction to such formulas. Students seem to regard (*) as *the statement being proved by induction*; that is, the context obscures the fact that a sequence of statements $S(n)$ must be proved where the n th statement $S(n)$ is (*). Another reason for student uncertainty with these arguments is the inherent chameleon role of the variable n : sometimes it is a fixed variable and sometimes it is a free variable.

When mathematicians read the above proof we mentally insert "Assume inductively for an arbitrarily fixed positive integer n that..." For us, the use of the variable in the principle of mathematical induction is clear. In the original statement n is a free variable. In the induction assumption and the subsequent algebra it is a fixed variable that is otherwise arbitrary. In the inductive conclusion it is again a free variable.

From the perspective of many of our students the situation is much less clear. For them, n usually denotes a free integer variable, in which case the inductive assumption is assuming what they are asked to prove. Nevertheless, they have been asked to prove something and so they continue. As evidence of this, we offer the following rather common phrasing of an induction argument that we see frequently in homework. (Students include more algebra for "justification").

Assume that:

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2} \text{ for all } n.$$

Then

$$1 + 2 + \cdots + n + (n+1) = \frac{n(n+1)}{2} + (n+1) = \frac{(n+1)(n+2)}{2} \text{ for all } n.$$

Thus, by the principle of mathematical induction the formula holds for all positive integers n .

In an effort to correct their understanding, we now ask our students to explicitly identify the sequence of statements to be proved and to use a different label to distinguish the free from the fixed roles of the variable. For example, the generic statement to be proved is identified as $S(n)$ and in the inductive step they assume $S(N)$ for a fixed but arbitrarily chosen positive integer N and then prove $S(N+1)$ follows.

In addition, we encourage them to summarize in words what they have just proved; namely, to write explicitly that the truth of $S(N)$ implies the truth of $S(N+1)$ before they invoke the principle of mathematical induction. When our students organize their proofs in this way, they typically say that N is fixed but they rarely mention that it is fixed arbitrarily. We take this to mean that they have an understanding of the importance of the fixed nature of N in the inductive hypothesis but do not yet fully appreciate the fact that the inductive argument must work from an arbitrary positive integer to the next one.

After listening to students in this course, we now see little advantage to beginning with problems that are as prescriptive as Gauss' formula and now prefer problems that allow students to work on the subtle aspects of induction in a less artificial setting. Elementary discrete mathematics has a wealth of such

problems, and two we find especially informative follow. The informative aspects of these problems include:

(A) Induction can be applied in many situations, not just to formulas involving n .

(B) The sequence of statements to be proved and the inductive proof benefit significantly from an essentially verbal statement of the sequence $S(n)$.

Example One (*The Sum Principle*) *If a finite set is partitioned into a finite number of blocks, then the sum of the sizes of the blocks in the partition is the size of the set.*

As students work on this problem, it usually helps to point out that the base case is the partition into two blocks, and the result is true by definition of addition. In order to carry out a reasonable proof by induction, a student must recognize to induct on the number of blocks in a partition (rather than the size of the set) and to express the general statement as:

$S(n)$: For any finite set and for any partition of the set into n blocks, the size of the set is the sum of the sizes of the blocks in the partition.

The proof of the inductive step must begin with an arbitrary partition of a set X into $N + 1$ blocks that is then related to an N -block partition of the same set X to which the induction hypothesis can be applied. Two essential features of this problem are realizing why the second “any” of the statement is critical to a correct induction argument and carefully invoking the base case during the induction step.

Our experience is that too many symbols get in the way. Some students formulate the general statement roughly as:

$$S(n) : \text{If } X = \bigcup_{k=1}^n B_k, \text{ then } |X| = \sum_{k=1}^n |B_k|.$$

Since this notation seems to force them to think in terms of one set and one partition, they cannot get started. This is an example of a common problem that can be corrected by using words rather than symbols.

Example Two (*Vertices and Edges of a Tree*) *In a finite tree with at least two vertices, the number of vertices is one greater than the number of edges.*

We ask our students to prove this statement by induction using two facts that they have already established: (1) every tree with at least two vertices has at least one vertex of degree one; (2) if a vertex of degree one and the incident edge (but not its other endpoint) are removed from a tree, the resulting graph is a tree. With these two pieces of information, a natural induction argument proves the sequence of statements:

$S(n)$: Any tree with n vertices has $n - 1$ edges.

There is only one tree with two vertices and $S(2)$ is therefore true. In order to advance the induction step, take any tree T with $N+1$ vertices. Using the two earlier results, T can be pruned to a tree T_0 with N vertices; by the induction hypothesis T_0 has $N-1$ edges. Reattaching the removed edge and vertex to T_0 reconstructs T and proves that T has $N+1$ vertices and N edges. An appeal to the principle of mathematical induction completes the proof.

This example reinforces aspects (A) and (B) for most students. It also uncovered an interesting student preference (possibly encouraged by the falling domino analogy) that made carrying out the proof more challenging than we had expected.

In their proofs of the inductive step, students instinctively began with a tree with N vertices to which they added an edge and a vertex of degree one and they then invariably asserted that the induction step had been advanced! They missed the subtle point that such an argument brings with it the obligation to show that all trees with $N+1$ vertices can be obtained in this way. (This can be justified from the two facts established earlier.) Even the best students missed this point.

Although with later classes we mentioned in a general context the extra obligation incurred by this use of induction, most students again ignored the gap in the reasoning noted above. This uncovers another subtlety in inductive arguments that requires more time and experience to appreciate. For the most part, this problem didn't occur in carefully written solutions to example one. That is, students don't build up partitions of $N+1$ blocks from ones with N blocks since most students who initially consider this approach realize that adding a block changes the underlying set. We have occasionally seen students construct the argument in this way, but usually the confusion comes from their using the same label for what are in fact two different sets.

Our teaching has benefited from our discussions, and we hope others who teach induction find some value in these observations from our recent experiences in teaching mathematical induction to math majors.

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