Inhomogeneous Diophantine approximation of irrationals with quasi-periodic continued fractions

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A real number θ is called a Hurwitzian number of order m if there exist a finite number of arithmetic progressions, $f_1(x), \ldots, f_R(x)$, of order at most m (and at least one has order m) such that

$$\theta = [b_0; b_1, \ldots, b_n, f_1(1), \ldots, f_R(1), f_1(2), \ldots, f_R(2), \ldots] \ ;$$

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$$\theta = [b_0; b_1, \ldots, b_n, \overline{f_1(i), \ldots, f_R(i)}\ ]_{i=1}^{\infty} .$$

A BETTER TITLE:

Inhomogeneous Diophantine approximation
of some Hurwitzian numbers

Some Examples of Hurwitzian numbers

Roger Cotes (1714) found the continued fraction expansion of e:

$$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, \ldots] = [2; \overline{1\,,\,2j\,,\,1}\,]_{j=1}^{\infty}\ .$$

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Euler (1737) proved this was true, and also for $s \geq 2$,

$$e^{1/s} = [\,\overline{1\,,\,2sj-(s+1)\,,\,1}\,]_{j=1}^{\infty}$$

and

$$\frac{e^{1/s}+1}{e^{1/s}-1} = [\,\overline{(4j+6)s}\,]_{j=1}^{\infty}\,.$$

In correspondence with Hermite, Stieljes (1905) described the continued fractions of $e^{2/k}$ for odd k:

$$e^2 = [7; \overline{3j-1,\, 1\,,\, 1\,,\, 3j\,,\, 12j+6}\,]_{j=1}^{\infty}\,;$$
$$e^{2/(2s+1)} = [\,\overline{1, f_1(j,s), f_2(j,s), f_3(j,s), 1}\,]_{j=1}^{\infty}$$

where each $f_i(s,j)$ is a (known) first-order arithmetic progression.

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Perron (1920s) generalized Euler's formula to all integers $k > 0$:

$$\frac{e^{2/k}+1}{e^{2/k}-1} = [\,\overline{(2i-1)k}\,]_{i=1}^{\infty}\,.$$

The Homogeneous Approximation Constant:

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Hurwitz proved for all θ, the inequality

$$\left|\theta - \frac{p}{q}\right| \le \frac{1}{\sqrt{5}q^2}$$

has infinitely many solutions in integers p, q; all $L(\theta) \le 1/\sqrt{5}$.

Homogeneous Approximation and Continued Fractions:

Let $\{(p_k, q_k)\}$ be the convergents for $\theta = [b_0; b_1, b_2, \ldots]$.
Then the quality of approximation of $q_k\theta$ to zero is

$$\|q_k\,\theta\| = |q_k\,\theta - p_k| = \frac{1}{|q_k|\;\mu_k(\theta)}$$

where $\mu_k(\theta) = [b_{k+1}; b_{k+2}, \ldots] + [0; b_k, \ldots, b_1]$.

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The reason for this is:
Lagrange's Theorem: If a/b is a rational number such that (a, b) is not a convergent to θ and $0 < |b| \leq q_{k+1}$ then

$$\|b\,\theta\| \geq \|q_k\,\theta\| > \|q_{k+1}\,\theta\|.$$

Inhomogeneous Approximation Constants:

For fixed irrational θ and $\phi \notin \mathbb{Z}\theta + \mathbb{Z}$, the
Inhomogeneous Approximation Constant for the pair θ, ϕ is:

$$M(\theta, \phi) = \liminf_{|q| \to \infty} |q| \, \|q\,\theta - \phi\| \, .$$

Inhomogeneous Approximation and Continued Fractions:

Early 1900s:
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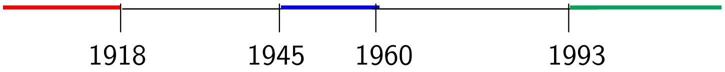

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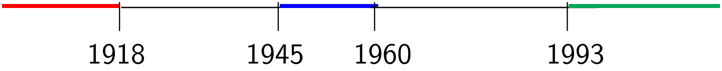

Mid 1900s:
Davenport, Ennola
Khintchine
Sós
Barnes, Swinnerton-Dyer
Pitman

Turn of the 21-st century:
Cusick, Moran, Pollington
Cusick, Rockett and Szüsz
Komatsu
Pinner

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In this talk, we show how the ideas of Grace and regular continued fractions can be used to unify and extend Komatsu's results.

For the remainder of the talk, we consider θ of the form

$$\theta = [a_0; c_1, \ldots, c_{n_1}, a_1, c_{1+n_1}, \ldots, c_{n_2}, a_2, \ldots]$$

where all $n_i \geq 0$, the c_j are bounded, and $\lim_{i \to \infty} a_i = \infty$.

And $\phi = \frac{r\,\theta + m}{n}$ is in "reduced form", namely,

$$\gcd(r, m, n) = 1\,,\ n \geq 2\,, \text{and}\, 0 \leq r, m < n\ .$$

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- The convergents

  $$P_i/Q_i := [a_0; c_1, \ldots, c_{n_1}, a_1, \ldots, c_{n_i}]$$

  will be called the leaping convergents.

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$$\gcd(r, m, n) = 1\,,\; n \geq 2\,, \text{and}\, 0 \leq r, m < n\,.$$

MAIN THEOREM: Suppose $M(\theta, \phi) < 1/n^2$. Then $M(\theta, \phi) = 0$ if and only if there exist infinitely many leaping convergents (P_j, Q_j) such that

$$g_j \, (P_j, Q_j) \equiv (m, -r) \pmod{n}$$

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An example:

For all $s \geq 2$, $e^{1/s} = [\,\overline{1\,,\,2sj-(s+1)\,,\,1}\,]_{j=1}^{\infty}$. Let's investigate $M(e^{1/s}, \phi)$ for reduced $\phi = (re^{1/s}+m)/2$.

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		1	s+1	1	1	s+1	1
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giving $M(e^{1/s},(\theta+1)/2) = 0$. Also,

		1	s+1	1	1	s+1	1
0	1	1	s	s+1	1	0	1
1	0	1	s+1	s	1	1	0

$$\mu_{3j-2}(e^{1/s}) = [1; 1, 2s(j+1) - (s+1), 1, \ldots] \\ +[0; 2sj - (s+1), 1, 1, 2s(j-1) - (s+1), 1, \ldots] \to 2\,,$$

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\begin{aligned}
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ETC, giving $0 < M(\theta,\phi) \leq 1/8$ for each of $\phi = 1/2$, $\phi = \theta/2$.

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<u>AND</u>

$$M(\theta, \phi) = \left(\limsup_i \{ \mu_i(\theta) \ : \ (p_i, q_i) \equiv (m, -r) \pmod{n} \} \right)^{-1}.$$

Therefore, $M(\theta, \phi) = 1/8$ for each of $\phi = 1/2$, $\phi = \theta/2$.

So, information on the congruence classes of convergents is needed.
Let's return to inspecting the convergents modulo $n ------>$

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- The modulo n sequences of convergents for $\theta = e^{2/k}$ are not anomalous for e and e^2.

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More about the leaping convergents of $e^{1/s}$. $- - - - >$

Computing the sequence of leapers for $e^{1/s}$.

C.S.Davis (1945): The leaping convergents of $e^{1/s}$ satisfy $(P_{-1}, Q_{-1}) = (1, -1)$, $(P_0, Q_0) = (1, 1)$, and

$$(P_{j+1}, Q_{j+1}) = A_j(P_j, Q_j) + (P_{j-1}, Q_{j-1}) \ ,$$

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where $A_j = 2(b_{k_j+1} + 1) = 2s(2j - 1)$.

Therefore the sequence of modulo n leaping convergents of $e^{1/s}$ is completely periodic for every n.

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Consequently, when n is odd whether or not $M(e^{1/s}, 1/n)$ is zero can be decided in $r = \lfloor n/2 \rfloor$ steps. Similarly for $M(e^{1/s}, \theta/n)$.

An indication of the proof:

The recurrence for the leapers of $e^{1/s}$ is:

$$(P_{j+1}, Q_{j+1}) = A_j(P_j, Q_j) + (P_{j-1}, Q_{j-1}) \ ,$$

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P_0	P_0	P_1	$\dots$	P_r	P_{r-1}	$\dots$	P_0	P_0	$\dots$
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D.N.Lehmer (1918), C.Elsner (1999), T.Komatsu (2004).