Balancing Gray codes

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Definitions

An n-digit Gray code is a sequence of n-strings using the alphabet $\{0, 1\}$ such that

- each n-string occurs exactly once;
- two consecutive strings differ in one digit.

Its transition sequence is the sequence of digit changes.

The code is cyclic if the last and first strings also differ in one digit.

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An example

EXAMPLE: R=2, n=3.

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Transition sequence (cyclic): 3,2,3,1,3,2,3,1

Transition counts: TC(1)=2 TC(2)=2 TC(3)=4

Transition spectrum: (2,2,4)

Can record as a Hamiltonian cycle in a Gray code graph.

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Spectrum for n-digit is $(2, 2, 4, \ldots, 2^{n-1})$

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Relative uniformity of transition-count distribution

A Gray code is called uniform if all transition counts are equal.
[The common value is $2^n/n$; n must be a power of 2.]

NOTE: Transition counts must be even.

A Gray code is balanced if every pair of transition counts satisfies

$$|TC(i) - TC(j)| \leq 2 \, .$$

Robinson & Cohn (1981), Knuth (1986, 2005),
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Definition

Fix integers $R, n \geq 2$.
An n-digit R-ary Gray code is a sequence of n-strings using the alphabet $\{0, 1, \ldots, R-1\}$ such that

- each n-string occurs exactly once;
- two consecutive strings differ in one digit and by ± 1.

It is cyclic if the last and first strings also differ in one digit by ± 1.

Its transition sequence is the sequence of digit changes. The code is uniform if all transition counts are equal (with common value of R^n/n).

A ternary example

EXAMPLE: R=3, n=2

00 10 11 01 02 12 22 21 20

Transition sequence: 1,2,1,2,1,1,2,2,1

Transition counts: TC(1)=5 TC(2)=4

Transition spectrum: (5,4)

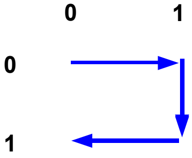
Next we give construction (Bose & Flahive 2007) of 2-digit R-ary codes whose spectrum is such that every transition count is either

$$\lfloor R^2/2 \rfloor \quad \text{or} \quad \lceil R^2/2 \rceil \, .$$

— — — — —— >

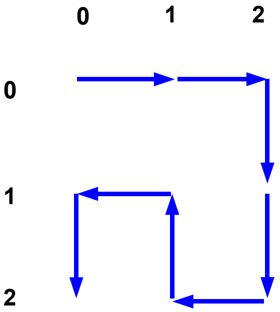
Base Cases: R=2, R=3

00 01 11 10



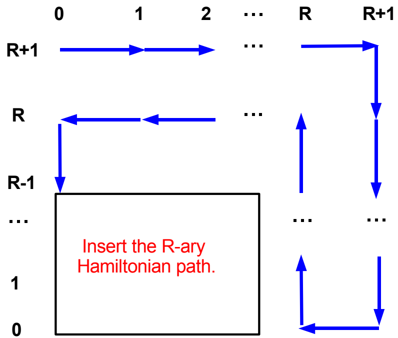
(2,2)

00 01 02 12 22 21 11 10 20



(4,5)

Inductive Step:



PROOF:

$$\lfloor R^2/2 \rfloor + 2R + 2 = \lfloor (R+2)^2/2 \rfloor \quad \text{and} \quad \lceil R^2/2 \rceil + 2R + 2 = \lceil (R+2)^2/2 \rceil$$

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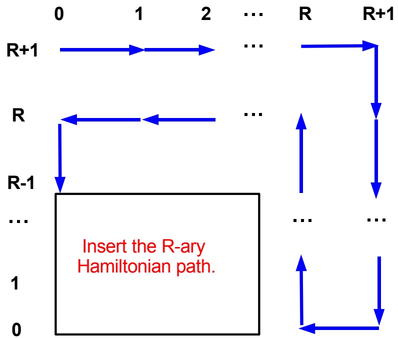

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How to extend to longer strings?

We call a code nearly-balanced if all its transition counts satisfy

$$\left| TC(j) - \frac{R^n}{n} \right| < \rho \quad \text{for all } 1 \leq j \leq n\,,$$

where ρ equals $R-1$ or $R-2$, whichever is even.

SOME REMARKS:

1. Our use of R^n/n as a reference point.
2. When R is even, every TC must be even.
3. For $R=4$ and $n=2^k$, then since $\rho=2$ every nearly-balanced code is uniform.

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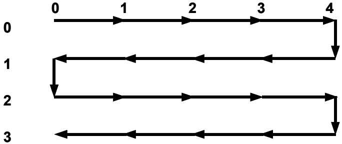
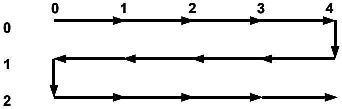
Basis for our construction:

We will use a basic Hamiltonian path on a rectangular grid:





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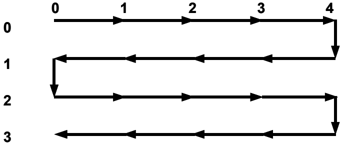

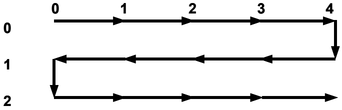

Note this is a *greedy* construction.

Outline of our construction:

STEP 1: Construct the graph. Label the rows of the $R^n \times R$ grid using a nearly-balanced (cyclic) n-digit code, and order the column labels by 0 1 . . . R-1.

STEP 2: Partition the row indices. Identify some (say L) elements of the transition sequence as connecting digits. This induces a partition on the n-digit code which has L blocks. Cycle the n-code as necessary to begin at the start of one of the blocks.

STEP 3: Construct a Hamiltonian path for an (n+1)-digit code. Let $M := R - 1, R$, whichever is odd. Use the basic construction on the $L \times M$ grid to construct a Hamiltonian path that "respects the partition-blocks"; that is: $- - -- >$

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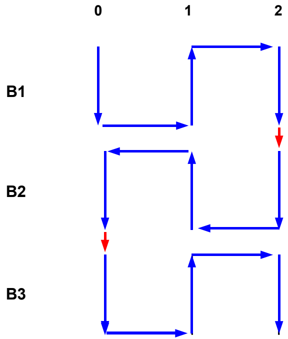
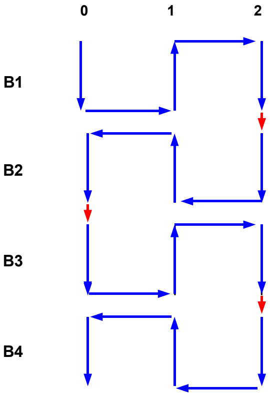
Hamiltonian paths that respect partition-blocks:

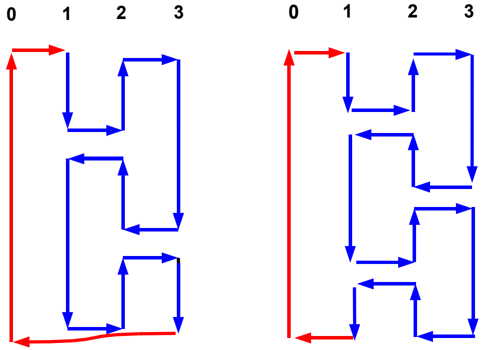




NOTE: An *odd* number of columns is needed.

Getting the Hamiltonian cycles

When R is even, this construction can be used to yield a Hamiltonian cycle regardless of the parity of partition-size:



The transition spectrum

We give the details for odd R.

Let $(t_1, \ldots, t_n)$ be the transition spectrum for any nearly-balanced n-digit code. For any choice of partition with connecting multiplicities $k_1, \ldots, k_n$, the transition spectrum of the (n+1)-digit code induced by the partition is:

$$(Rt_1 - k_1\rho \ , \ldots , \ Rt_n - k_n\rho \ , \ L\rho),$$

where $L := \sum k_j$ is the number of partition-blocks.

<u>CAUTION:</u> In order to get cyclic, L must be even.

Finding a good set of connecting digits:

For each $j = 1, \ldots, n$, let k_j be the integer such that

$$0 \leq R\, t_j - \frac{R^{n+1}}{n+1} - k_j\, \rho < \rho\,.$$

(Denote these deviation values by S_j.)

It can be shown that

$$(t_1, \ldots, t_n) \text{ nearly balanced} \implies \text{all } 0 \leq k_j < t_j\,.$$

Use these as trial connecting multiplicities.

What about the size of $L\,\rho$? Is it large enough?

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Finding a good partition, continued:

$L\,\rho$ is usually not large enough, but remember that $k_j < t_j$:

Recall $S_j := R\,t_j - k_j\,\rho - \frac{R^{n+1}}{n+1}$, and define $M := \lfloor \sum S_j/\rho \rfloor$.

Since

$$\sum S_j = R \sum t_j - L\,\rho - n\frac{R^{n+1}}{n+1} = \frac{R^{n+1}}{n+1} - L\,\rho\,;$$

$$0 \leq \frac{R^{n+1}}{n+1} - (L+M)\,\rho < \rho\,.$$

Parity arguments show the LHS equality cannot occur. Therefore, for either L+M or L+M+1 partition-blocks (whichever is even) the transition spectrum will be nearly-balanced.

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We then prove at least M+1 of the S_i are nonzero, and so increasing either M or M+1 of the k_j by 1 gives the result.

An observation

For $R = 3, 4$, nearly balanced means

$$\left| TC(j) - \frac{R^n}{n} \right| < 2\,.$$

- $R = 4$: TCs even, and so get balanced.
- $R = 3$: 3 or 4 different transition counts. Which can be reduced to 2 or 3 by a trick.

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