$h : H^n(\mathcal{F}; G) \to \mathrm{Hom}(H_n, G)$

Lemma: Given a chain complex $\mathcal{F}$ and an abelian group G, there is a natural homomorphism

$$h : H^n(\mathcal{F}; G) \to \mathrm{Hom}(H_n, G)$$

that is surjective.

Kernel of Surjection

Lemma: There is a split exact sequence:

$$0 \to \ker(h) \to H^n(\mathcal{F}; G) \xrightarrow{h} \mathrm{Hom}(H_n, G) \to 0$$

Details

Lemma: $\ker(h) \simeq \mathrm{Ext}(H_{n-1}, G)$.

Main Idea: Analyze the diagrams:

$$\begin{array}{ccccccccc}
0 & \to & Z_{n+1} & \to & C_{n+1} & \xrightarrow{\partial} & B_n & \to & 0 \\
 & & \downarrow 0 & & \downarrow \partial & & \downarrow 0 & & \\
0 & \to & Z_n & \to & C_n & \xrightarrow{\partial} & B_{n-1} & \to & 0
\end{array}$$

$$\begin{array}{ccccccccc}
0 & \leftarrow & Z^*_{n+1} & \leftarrow & C^*_{n+1} & \xleftarrow{\tilde{\partial}} & B^*_n & \leftarrow & 0 \\
 & & \uparrow 0 & & \tilde{\partial} \uparrow & & \uparrow 0 & & \\
0 & \leftarrow & Z^*_n & \leftarrow & C^*_n & \xleftarrow{\tilde{\partial}} & B^*_{n-1} & \leftarrow & 0
\end{array}$$

and $\quad 0 \to B_{n-1} \xrightarrow{i} Z_{n-1} \to H_{n-1} \to 0$

Universal Coefficient Theorem

Theorem 3.2: If a chain complex $\mathcal{F}$ of free abelian groups has homology groups H_n, then the cohomology groups $H^n(\mathcal{F}; G)$ of the cochain complex $\mathrm{Hom}(F_n, G)$ are determined by split exact sequences

$$0 \to \mathrm{Ext}(H_{n-1}, G) \to H^n(\mathcal{F}; G) \to \mathrm{Hom}(H_n, G) \to 0$$

Corollary 3.3: If H_n and H_{n-1} are finitely generated, then $H^n(\mathcal{F}; \mathbb{Z}) \simeq H_n / T_n \oplus T_{n-1}$

Corollary 3.4: If a chain map between chain complexes of free abelian groups induces homology isomorphisms, then it also induces cohomology isomorphisms for any G.