Tensor Products

Definition: If A and B are abelian groups, $F(A,B)$ is the free abelian group generated by the set $A \times B$, and $R(A,B)$ is the subgroup generated by elements of the form:

$$(a+a',b)-(a,b)-(a',b) \text{ and } (a,b+b')-(a,b)-(a,b'),$$

then $A \otimes B$ is defined to be

$$F(A,B)/R(A,B)$$

Properties

- $(a+a') \otimes b = a \otimes b + a' \otimes b$
- $a \otimes (b+b') = (a \otimes b) + (a \otimes b')$
- $a \otimes 0 = 0 \otimes b = 0$
- $na \otimes b = n(a \otimes b) = a \otimes nb$.
- Bilinear functions from $A \times B$ to G induce homomorphisms from $A \otimes B \longrightarrow G$.
- If $f : A \longrightarrow A'$ and $g : B \longrightarrow B'$ are homomorphisms, there is a unique homomorphism $f \otimes g : A \otimes B \longrightarrow A' \otimes B'$ such that $(f \otimes g)(a \otimes b) = f(a) \otimes g(b)$.

Other Results

Theorem:
$Z \otimes G \simeq G$ by an isomorphism that takes $n \otimes g$ to ng.

Theorem:
If $f : A \longrightarrow A'$ and $g : B \longrightarrow B'$ are surjective, then $f \otimes g : A \otimes B \longrightarrow A' \otimes B'$ is also surjective, and has kernel the subgroup of $A \otimes B$ generated by all elements of the form $a \otimes b$
for which $a \in \ker(f)$ or $b \in \ker(f)$.

Exactness

Theorem:

If $A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is exact, then so is

$$A \otimes G \xrightarrow{f \otimes i} B \otimes G \xrightarrow{g \otimes i} C \otimes G \longrightarrow 0$$

If f is injective and the first sequence splits, then $f \otimes i$ is injective and the second sequence spits.

If G is torsion free and f is injective, then $f \otimes i$ is injective.

Computations

Corollary: There is a natural iso. $Z_m \otimes G \simeq G/mG$

Theorem:

(a) $A \otimes B \simeq B \otimes A$

(b) $\oplus A_\alpha \otimes B \simeq \oplus (A_\alpha \otimes B)$

(c) $(A \otimes B) \otimes C \simeq A \otimes (B \otimes C)$

Theorem:
If A is free abelian with basis $\{a_i\}$ and B is free abelian with basis $\{b_j\}$ then $A \otimes B$ is free abelian with basis $\{a_i \otimes b_j\}$

Torsion Products

Definition: Let A be an abelian group, $F(A)$ be the free abelian group generated by the elements of A, $R(A)$ denote the kernel of projection $F(A) \to A$, and let $0 \to R(A) \xrightarrow{\varphi} F(A) \to A \to 0$
be the canonical free resolution of A.

The group $Tor(A, B) = A * B$ is defined to $\ker(\varphi \otimes id) : R(A) \otimes B \to F(A) \otimes B$.

Uniqueness:

Definition: If $\gamma : A \to A'$ and $\delta : B' \to B$ are homomorphisms, and if we extend γ to a homomorphism : $F(A) \to F(A')$, then $\gamma * \delta : A * B \to A' * B'$ is defined to be the homomorphism induced by γ and δ relative to these free resolutions.

Theorem:
$A * B$ can be computed using any free resolution.

Naturality

Theorem There is a function that assigns to each free resolution $0 \to R \xrightarrow{\varphi} F \xrightarrow{\psi} A \to 0$ of A, and to each abelian group B, a natural exact sequence
$0 \to A * B \to R \otimes B \xrightarrow{\phi \otimes id} F \otimes B \xrightarrow{\psi \otimes id} A \otimes B \to 0.$

This function is natural in the sense that a homomorphism of free resolutions and a homomorphism of abelian groups gives rise to a homomorphism of exact sequences.

Computations

Theorem

(a) There is a natural isomorphism $A * B \simeq B * A$

(b) $\oplus A_\alpha * B \simeq \oplus(A_\alpha * B)$

(c) $A * B = 0$ if A is torsion free.

(d) Given B, there is an exact sequence
$$0 \to Z_m * B \to B \xrightarrow{\cdot m} B \to Z_m \otimes B \to 0$$

$\mathbb{Z} \otimes G \simeq G$ $\qquad$ $\mathbb{Z} * G \simeq 0$

$\mathbb{Z}_m \otimes G \simeq G/mG$ $\qquad$ $\mathbb{Z}_m * G \simeq \ker(G \xrightarrow{\cdot m} G)$

$\mathbb{Z}_m \otimes \mathbb{Z} \simeq \mathbb{Z}_m$ $\qquad$ $\mathbb{Z}_m * \mathbb{Z} \simeq 0$

$\mathbb{Z}_m \otimes \mathbb{Z}_n \simeq \mathbb{Z}_d$ $\qquad$ $\mathbb{Z}_m * \mathbb{Z}_n \simeq \mathbb{Z}_d$