Poincare Duality

Theorem: If M is a closed, R-orientable n-manifold with fundamental class $\Lambda \in H_n(M;R)$, then the map $D : H^k(M;R) \longrightarrow H_{n-k}(M;R)$ defined by $D(\alpha) = \Lambda \frown \alpha$ is in isomorphism for all k.

Noncompact Manifolds

Definition: There is a duality map $D_M : H_c^k(M;R) \longrightarrow H_{n-k}(M;R)$ for arbitrary R-orientable manifolds defined taking limits in:

$$\begin{array}{ccccc} H_n(M|L;R) & \times & H^k(M|L;R) & \xrightarrow{\frown} & \\ \Big\downarrow{\scriptstyle i_*} & & \Big\uparrow{\scriptstyle i^*} & & H_{n-k}(M;R) \\ H_n(M|K;R) & \times & H^k(M|K;R) & \xrightarrow{\frown} & \end{array}$$

Poincare Duality Theorem II: The duality map $D_M : H_c^k(M;R) \longrightarrow H_{n-k}(M;R)$ is an isomorphism for all k, whenever M is an R-oriented n-manifold.

Steps in Proof

Lemma: If M is the union of open sets U and V, then there is a diagram of M-V sequences, commutative up to sign:

$$\begin{array}{ccccccc} H_c^k(U\cap V) & \to & H_c^k(U)\oplus H_c^k(V) & \to & H_c^k(M) & \to & H_c^{k+1}(U\cap V) \\ \Big\downarrow{\scriptstyle D_{U\cap V}} & & \Big\downarrow{\scriptstyle D_U\oplus -D_V} & & \Big\downarrow{\scriptstyle D_M} & & \Big\downarrow{\scriptstyle D_{U\cap V}} \\ H_j(U\cap V) & \longrightarrow & H_j(U)\oplus H_j(V) & \longrightarrow & H_j(M) & \longrightarrow & H_{j-1}(U\cap V) \end{array}$$

where $j = n - k$

Steps in Proof, Continued

- M is the union of open sets U and V, and if D_U, D_V, and $D_{U\cap V}$ are isomorphisms, so is D_M
- M is the union of $U_1 \subset U_2 \subset \cdots$ and if each D_{U_i} is an isomorphism, so is D_M
- $M = R^n$
- M is an open set in R^n
- M is a finite or countable union of open sets U_i each homeomorphic to R^n