Def: A space X is universal for a class of spaces C if X ∈ C and if each element of C is homeomorphic to a subspace of X.

Thm: The spaces $\prod_{i=1}^{\infty}\{0,1\}$ and $\prod_{i=1}^{\infty} \mathbf{Z}_+$ are universal spaces for the class of zero-dimensional spaces.

Lemma: The Cantor set is homeomorphic to $\prod_{i=1}^{\infty}\{0,1\}$ and the space of irrationals is homeomorphic to $\prod_{i=1}^{\infty} \mathbf{Z}_+$.

Cor: The Cantor set and the space of irrationals are universal spaces or the class of zero-dimensional spaces.

Steps in showing $\prod_{i=1}^{\infty} \mathbf{Z}_+ \cong \mathbf{R} - \mathbf{Q} = \mathbf{P}$:

(Exercise)

- **Def:** A space is completely metrizable if it has an equivalent metric that is complete.
- Closed subspaces and countable products of completely metrizable spaces are completely metrizable.
- G_δ subspaces of completely metrizable spaces are completely metrizable.
- P is completely metrizable.
- ∀ metric ρ on P, $\forall \epsilon > 0$, and for every nonempty open subset U of P, ∃ a sequence $F_1, F_2, \ldots$ of pairwise disjoint clopen subsets whose union is U and whose diameters are $< \epsilon$
- Use the previous step to associate with each $x \in P$ a sequence of natural numbers and show that this association is a $\cong$.

Compactification Thm: Every zero dimensional space X has a zero dimensional compactification X′.

Embedding Thm: Every zero dimensional space X is homeomorphic to a subspace of $\mathbf{R} = \mathbf{R}^{2\cdot 0+1}$.

Types of Disconnectedness

Consider the following properties:

(1) X is totally disconnected, i.e. the components of X are single points.
(2) Any two points in X can be separated by the empty set.
(3) Any point in X can be separated from a closed set not containing it by the empty set (i.e. X is zero dimensional).
(4) Any two disjoint closed sets in X can be separated by the empty set.

We have: $4 \iff 3 \Rightarrow 2 \Rightarrow 1$.

Lemma: If X is compact, $3 \iff 2$

Lemma: If X is compact, $p \in X$ and $M(p)$ is the set of points which cannot be separated from p by the empty set, then $M(p)$ is connected.

Lemma: If X is compact, $2 \iff 1$

Thm: If X is compact, then zero dimensionality is equivalent to total disconnectedness, i.e., all four properties listed previously are equivalent.

Example: There is a one dimensional space that is totally disconnected.

Exercise:
C = Cantor set. Let Q be the endpoints of removed intervals in C and let P be the remaining points.
For each $x \in Q$, let L_x be the set of points on the segment joining x to c $= (\mathbf{1/2}, \mathbf{1})$ with rational second coordinate. For each $x \in P$, let L_x be the set of points on the segment joining x to c with irrational second coordinate.
Let T be the union of the $\mathbf{L_x}$.
Claim: T is connected, so not zero dimensional.
So $\mathbf{T} - \mathbf{c}$ is not zero dimensional. $\mathbf{T} - \mathbf{c}$ is totally disconnected.