Mth 676 - Outline of Results:
All spaces under consideration are separable metric.

Def.

- $\text{ind}(X) = -1$ iff $X = \emptyset$
- $\text{ind}(X) \leq n$ if $\forall x$ and for each nbhd V of x, there is a nbhd U of x with $U \subset V$ and with $\text{ind}(Bd(U)) \leq n-1$
- $\text{ind}(X) = n$ if $\text{ind}(X) \leq n$ and $\text{ind}(X) > n-1$
- $\text{ind}(X) = \infty$ if $\text{ind}(X) > n$ for all n.

Theorem: $\forall$ subspace M of X, $\text{ind}(M) \leq \text{ind}(X)$

Def. L is a partition between disjoint subspaces A and B of X if

Theorem: X satisfies the $\text{ind}(X) \leq n$ iff $\forall x \in X$ and for each closed B with $x \notin B$, there exists a partition L between $\{x\}$ and B with $\text{ind}(L) \leq n-1$.

Thm. ind$(X) \leq n$ iff X has a countable basis $\{U_i\}$ with ind$(Bd(U_i)) \leq n-1$ for each i.

Dimension Zero

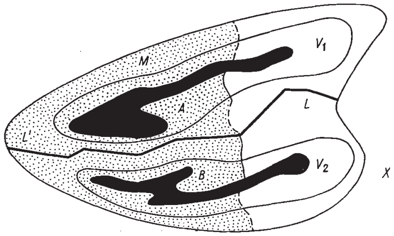
Thm: ind$(X) = 0$
iff $X \neq \emptyset$ and for each x in X and nbhd V of x there exists a clopen U with $x \in U \subset V$
iff X has a countable basis consisting of clopen sets
iff for each $x \in X$ and closed set B not containing x, the empty set is a partition between $\{x\}$ and B.

Thm: Every non empty subspace of a space X with ind$(X) = 0$ has ind$= 0$.

Thm: **The first separation theorem for dimension 0.** If X is a zero-dimensional pace, then for every pair A, B of disjoint closed subsets of X the empty set is a partition between A and B.

Recall, for each pair of separated sets A and B in X, there are disjoint open sets U and V with $A \subset U$ and $B \subset V$.

Lemma: Let M be a subspace X and A, B a pair of disjoint closed subsets of X. Let V_1, V_2 be disjoint opens subsets containing A and B respectivly, with disjoint closures. For every partition L' in the space M between $M \cap \overline{V_1}$ and $M \cap \overline{V_2}$, there exists a partition L in X between A and B with $M \cap L \subset L'$



If M is closed, For every partition L' in the space M between $M \cap A$ and $M \cap B$, there exists a partition L in X between A and B with $M \cap L \subset L'$

Thm: Second Separation Theorem for Dimension Zero.
If Z is a zero-dimensional subspace of X , then for each pair of disjoint closed subsets A and B if X there exists a partition L between A and b with $L \cap Z = \emptyset$.

Cor: A subspace M of X has ind(M)= 0
iff for each $x \in M$ and each nbhd V of x in X there is an open $U \subset X$, $x \in U$, with $\mathrm{Bd}(U) \cap M = \emptyset$
iff X has a ctble basis $\{U_i\}$ with $\mathrm{bd}(U_i) \cap M = \emptyset$ for each i.

Thm: Enlargement Thm for Dimension Zero For each zero dimensional $Z \subset X$ there is a G_δ set $Z_* \subset X$ with $Z \subset Z_*$ and ind$(Z_*)= 0$.

Thm: Sum Theorem for Dimension Zero

If $X = \cup_{i=1}^{\infty} F_i$ where the F_i are closed in X and ind$(F_i)= 0$, then ind$(X)= 0$.

Cor: If $X = \cup_{i=1}^{\infty} F_i$ where the F_i are F_σ sets in X and ind$(F_i)= 0$, then ind$(X)= 0$.

Cor: If X is the union of zero dimensional subpaces A and B where A is closed in X, then ind$(X)= 0$.

Cor: If X is the union of a zero dimensional subpace A and a finite set B, then ind$(X)= 0$.

Thm: Product Theorem for Dimension Zero

Let $X = \prod_{i=1}^{\infty} X_i$. Then ind($X$)= 0 if and only if ind($X_i$)= 0 for each i.

Cor: The inverse limit of a sequence of zero dimensional spaces is either empty or of dimension zero.

Example: The subspaces $Q^n_k \subset R^n$ consisting of points with exactly k rational coordinates are zero dimensional.