# CONTRACTIBLE 3-MANIFOLDS AND THE DOUBLE 3-SPACE PROPERTY

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ABSTRACT. Gabai showed that the Whitehead manifold is the union of two submanifolds each of which is homeomorphic to $\mathbb{R}^3$ and whose intersection is again homeomorphic to $\mathbb{R}^3$. Using a family of generalizations of the Whitehead Link, we show that there are uncountably many contractible 3-manifolds with this double 3-space property. Using a separate family of generalizations of the Whitehead Link and using an extension of interlacing theory, we also show that there are uncountably many contractible 3-manifolds that fail to have this property.

## 1. Introduction

Gabai [Gab11] showed a surprising result that the Whitehead contractible 3-manifold [Whi35] is the union of two sub-manifolds each of which is homeomorphic to Euclidean 3-space $\mathbb{R}^3$ and whose intersection is also homeomorphic to $\mathbb{R}^3$. A 3-manifold with this *double 3-space property* must be a contractible open 3-manifold. The manifold $\mathbb{R}^3$ clearly has this property, but it takes a lot of ingenuity to show that the Whitehead contractible 3-manifold has the double 3-space property. This naturally raises two questions:

1) Are there other contractible 3-manifolds with this property?
2) Do all contractible 3-manifolds have this property?

We show the answer to the first question is yes by constructing uncountably many contractible 3-manifolds with the double 3-space property. We show the answer to the second question is no by constructing uncountably many contractible 3-manifolds that fail to have the double 3-space property. The answer to the second question requires a careful extension of interlacing theory originally introduced in [Wri89].

## 2. Definitions and Preliminaries

A solid torus is homeomorphic to $B^2 \times S^1$ where $B^2$ is a 2-dimensional disk and $S^1$ is a circle. All spaces and embeddings will be piecewise-linear [RS82]. If $M$ is a manifold with boundary, then Int $M$ denotes the interior of $M$ and $\partial M$ denotes the boundary of $M$. We let $\mathbb{R}^3$ denote Euclidean 3 -space. A *disk with holes* is a compact, connected planar 2-manifold with boundary. A properly embedded disk with holes $H$ in a solid torus $T$ is said to be

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*interior-inessential* if the inclusion map on $\partial H$ can be extended to a map of $H$ into $\partial T$. If the inclusion map on $\partial H$ cannot be extended to a map of $H$ into $\partial T$ we say that $H$ is *interior-essential* [Dav07], [DV09, p. 170]. If $H$ is interior-essential, we also say $H$ is a *meridional disk with holes* for the solid torus $T$.

For background on contractible open 3-manifolds, see [McM62, Mye88, Mye99, Wri92].

**Definition 2.1.** A *Whitehead Link* is a pair of solid tori $T' \subset \text{Int } T$ so that $T'$ is contained in Int $T$ as illustrated in Figure 1(a).

The famous Whitehead contractible 3-manifold [Whi35] is a 3-manifold that is the ascending union of nested solid tori $T_i, i \geq 0$, so that for each $i$, $T_i \subset \text{Int } T_{i+1}$ is a Whitehead Link.

**Definition 2.2.** If $T' \subset \text{Int } T$ are solid tori, the *geometric index* of $T'$ in $T$, $N(T', T)$, is the minimal number of points of the intersection of the centerline of $T'$ with a meridional disk of $T$.

**Note**: If $T' \subset \text{Int } T$ is a Whitehead Link, then the geometric index of $T'$ in $T$ is 2.

See Schubert [Sch53] and [GRWŽ11] for the following results about geometric index.

- Let $T_0$ and $T_1$ be unknotted solid tori in $S^3$ with $T_0 \subset \text{Int } T_1$ and $N(T_0, T_1) = 1$. Then $\partial T_0$ and $\partial T_1$ are parallel.
- Let $T_0, T_1$, and $T_2$ be solid tori so that $T_0 \subset \text{Int } T_1$ and $T_1 \subset \text{Int } T_2$. Then $N(T_0, T_2) = N(T_0, T_1) \cdot N(T_1, T_2)$.

We now define a generalization of the Whitehead Link (which has geometric index 2) to a *Gabai Link* that has geometric index $2n$ for some positive integer $n$. We will use this generalization in Section 3 to produce our examples of 3-manifolds that have the double 3-space property.

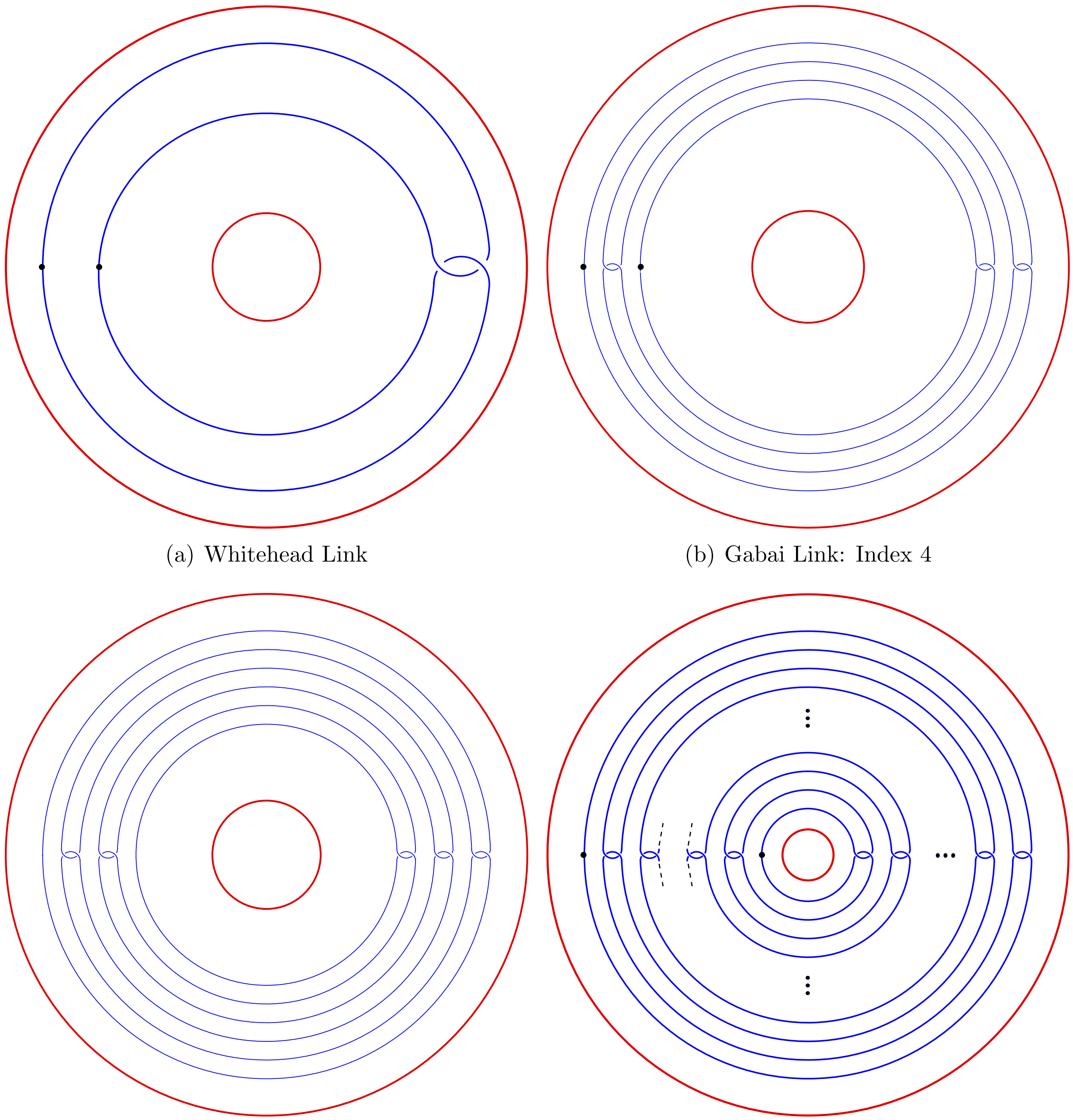
**Definition 2.3.** Let $n$ be a positive integer. A *Gabai Link* of geometric index $2n$ is a pair of solid tori $T' \subset \text{Int } T$ as illustrated in Figure 1. Figure 1(b) shows a Gabai Link of index 4, Figure 1(c) shows a Gabai Link of index 6, and Figure 1(d) shows a generalized Gabai Link of index $2n$. For the link of geometric index $2n$, there are $n - 1$ clasps on the left and $n$ clasps on the right.

Note that the inner torus $T'$ in a Gabai Link is contractible in the outer torus $T$.

**Definition 2.4.** A *genus one 3-manifold* $M$ is the ascending union of solid tori $T_i, i \geq 0$, so that for each $i$, $T_i \subset \text{Int } T_{i+1}$ and the geometric index of $T_i$ in $T_{i+1}$ is not equal to 0.

**Theorem 2.5.** *If $M$ is a genus one 3-manifold with defining sequence $(T_i)$, then, for each $j$, $T_j$ does not lie in any open subset of $M$ that is homeomorphic to $\mathbb{R}^3$.*

Proof. If $T_j$ lies in $U$ so that $U$ is homeomorphic to $\mathbb{R}^3$, then, since $T_j$ is compact, it lies in a 3-ball $B \subset U$. Since $B$ is compact, it lies in the interior of some $T_k$ with $k > j$. This implies that the geometric index of $T_j$ in $T_k$ is 0, but since the geometric index is multiplicative, the geometric index of $T_j$ in $T_k$ is not zero. So there is no such $U$. $\square$



(a) Whitehead Link

(b) Gabai Link: Index 4

(c) Gabai Link: Index 6

(d) Gabai Link: Index 2n

FIGURE 1. Whitehead and Gabai Links

**Theorem 2.6.** *If $M$ is a genus one 3-manifold with defining sequence $(T_i)$, and $J$ is an essential simple closed curve that lies in some $T_j$, then $J$ does not lie in any open subset of $M$ that is homeomorphic to $\mathbb{R}^3$.*

PROOF. By thickening up $T_j$ we may assume, without loss of generality, that $J$ is the centerline of a solid torus $T$ that lies in Int $T_j$. Since $J$ is essential in $T_j$, the geometric index of $T$ in $T_j$ is not equal to zero. Thus, $M$ is the ascending union of tori $T, T_j, T_{j+1}, T_{j+2}, \cdots$

and by the previous theorem, $T$ does not lie in any open subset of $M$ that is homeomorphic to $\mathbb{R}^3$. If $J$ lies in $U$ so that $U$ is homeomorphic to $\mathbb{R}^3$, then we could have chosen $T$ so that it also lies in $U$. Thus, by Theorem 2.5, $J$ does not lie in any open subset of $M$ that is homeomorphic to $\mathbb{R}^3$. □

**Theorem 2.7.** *A genus one 3-manifold $M$ with defining sequence $(T_i)$ so that each $T_i$ is contractible in $T_{i+1}$, is a contractible 3-manifold that is not homeomorphic to $\mathbb{R}^3$.*

PROOF. It is contractible since all the homotopy groups are trivial. If $M$ is homeomorphic to $\mathbb{R}^3$, then each $T_i$ in the defining sequence lies in an open subset that is homeomorphic to $\mathbb{R}^3$ which is a contradiction. □

**Definition 2.8.** A 3-manifold is said to satisfy the *double 3-space property* if it is the union of two open sets $U$ and $V$ so that each of $U$, $V$, and $U \cap V$ is homeomorphic to $\mathbb{R}^3$.

## 3. Gabai Manifolds Satisfy the Double 3-space Property

3.1. **Gabai Manifolds.** Refer to Definition 2.3 and Figure 1 for the definition of a Gabai Link.

**Definition 3.1.** A *Gabai contractible 3-manifold* is the ascending union of nested solid tori so that any two consecutive tori form a Gabai Link. Given a sequence $n_1, n_2, n_3, \ldots$ of positive integers, there is a Gabai contractible 3-manifold $G = \bigcup_{m=0}^{\infty} T_m$ so that the tori $T_{m-1} \subset \text{Int } T_m$ form a Gabai Link of index $2\,n_m$.

In fact, it is possible to assume that each $T_m \subset \mathbb{R}^3$ because if a Gabai Link is embedded in $\mathbb{R}^3$ so that the larger solid torus is unknotted, then the smaller solid torus is also unknotted. McMillan's proof [McM62] that there are uncountably many genus one contractible 3-manifolds transfers immediately to show that there are uncountably many Gabai contractible 3-manifolds. This proof uses properties of geometric index to show that if a prime $p$ is a factor of infinitely many of $n_1, n_2, n_3, \ldots$ and only finitely many of $m_1, m_2, m_3, \ldots$, then the two 3-manifolds formed using these sequences cannot be homeomorphic.

3.2. **Special Subsets of $S^1$ and $B^2 \times S^1$.** In $S^1$ choose a closed interval $I$ which we identify with the closed interval $[0, 1]$. Let $C \subset I \subset S^1$ be the standard middle thirds Cantor set. Let $U_1 = (\frac{1}{3}, \frac{2}{3})$, $U_2 = (\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9})$, and, in general, $U_i$ be the union of the $2^{i-1}$ components of $[0, 1] - C$ that have length $1/3^i$. Let $U_0 = S^1 - [0, 1]$, $C_1 = C \cap [0, \frac{1}{3}]$, and $C_2 = C \cap [\frac{2}{3}, 1]$.

Let $h : B^2 \times S^1 \to \mathbb{R}^3$ be an embedding so that $T = h(B^2 \times S^1)$ is a standard unknotted solid torus in $\mathbb{R}^3$. Set $V^i = h(B^2 \times U_i)$, $A = h(B^2 \times C_1)$, and $B = h(B^2 \times C_2)$. So $V^i$ (for $i \geq 0$), $A$, and $B$ are all subsets of $T$. The subset $V^0$ is homeomorphic to $B^2 \times (0, 1)$. For $i > 0$, $V^i$ is homeomorphic to the disjoint union of $2^{i-1}$ copies of $B^2 \times (0, 1)$, and both $A$ and $B$ are homeomorphic to $B^2 \times C$.

For each positive integer $n$, let $g_n$ be a homeomorphism of $\mathbb{R}^3$ to $\mathbb{R}^3$ that takes $T$ into its interior, so that the pair $(g_n(T), T)$ forms a Gabai Link of geometric index $2n$. Let $T_n' = g_n(T)$, $A_n' = g_n(A)$, $B_n' = g_n(B)$, and ${V_n^i}' = g_n(V^i)$.

**Lemma 3.2.** *The homeomorphisms $g_n : \mathbb{R}^3 \to \mathbb{R}^3$ can be chosen so that:*

$$A \cap T_n' = A_n' \text{ and } B \cap T_n' = B_n' \tag{1a}$$

$$V_n^{0\,\prime} \subset V^0 \text{ and for } i > 0, V_n^{i\,\prime} \subset V^j, \text{ where } j < i. \tag{1b}$$

Proof. Fix a positive integer $n$. We first define $g_n$ on $T$. The idea is to identify $4n$ subsets of $T = B^2 \times S^1$, each homeomorphic to a tube of the form an interval cross $B^2$, and to use the $S^1$ coordinate to linearly (in the $S^1 - U_0 = I$ factor) stretch these tubes from the region $V^0$ to the region $V^1$ in $T$.

Choose a positive integer $m > 0$ and a nonnegative integer $k < 2^{m-1}$ so that $2^m + 2k = 4n < 2^{m+1}$. Remove the subsets $U_1, \ldots U_m$ from $I$ so that $2^m$ intervals of length $3^{-m}$ remain. Then remove $2k$ of the intervals in $U_{m+1}$, namely the middle third of the first $k$ and the last $k$ of these remaining intervals in $I$ so that $4n = 2^m + 2k$ intervals remain, $4k$ of length $3^{-(m+1)}$, and the remaining $2^m - 2k$ of length $3^{-m}$. Let $\widetilde{U}_{m+1}$ be the union of the intervals in $U_{m+1}$ that have been removed. Figure 2 shows the case where $n = 3, m = 3,$ and $k = 2$. The integers $i$ across the bottom of this figure correspond to the $U_i$ defined above.


Figure 2. Labelled Removed Intervals in $[0, 1]$

Now let $\widetilde{V}_{m+1} = h(B^2 \times \widetilde{U}_{m+1})$ and consider $W = T - \cup_{j=0}^{m} V^j - \widetilde{V}_{m+1}$. $W$ consists of $4n$ tubes homeomorphic to an interval cross $B^2$. Let $g_n$ be a homeomorphism of $T$ into its interior so that:

(1) The pair $(g_n(T), T)$ forms a Gabai Link of geometric index $2n$,
(2) $g_n(V^0 \cup V^1) \subset V^0$,
(3) The components of $\cup_{j=2}^{m} V^j \cup \widetilde{V}_{m+1}$ are taken by $g_n$ into $V_0$ or $V_1$, and
(4) $g_n$ restricted to each of the $4n$ tubes mentioned above is a product of a homeomorphism of the $B^2$ factor onto a subdisk with a linear homeomorphism on the interval factor that stretches the tube from $V^0$ to $V^1$ or from $V^1$ to $V^0$ in either $B^2 \times [0, 1/3]$ or in $B^2 \times [2/3, 1]$.

Figure 3 illustrates this when $n = 3$, with the numbers $j$ listed by parts of the interior torus corresponding to the subsets $g_n(V^j)$. The last two regions mentioned in (4) above correspond to the top or bottom parts of the $T - (V_0 \cup V_1)$ in Figure 3. In particular, the $S^1$ factor, after $U_0$ is removed is parameterized in a counterclockwise manner in Figure 3.

The interval factor of each of the tubes in W corresponds to an interval in $I$ of length $3^{-m}$ or of length $3^{-(m+1)}$, one of the remaining intervals in stage $m$ or stage $m + 1$ of the standard construction of $C$. Let $D$ be one of these intervals. The self similarity of $C$ shows that a linear homeomorphism from $D$ onto either $E = [0, 1/3]$ or onto $E = [2/3, 1]$ takes $C \cap D$ onto $C \cap E$ and takes the intervals of $U_i \cap D$ homeomorphically to the intervals of $U_{i-k} \cap E$ where $k = m - 1$ or $k = m$.

From this, it follows that condition (1b) is satisfied. The nature of a Gabai Link guarantees that $A_n' \subset B^2 \times [0, 1/3] \subset T$ and that $B_n' \subset B^2 \times [1/3, 2/3] \subset T$. This, together with the discussion in the previous paragraph shows that condition (1a) is satisfied.

Since both $T$ and $T' = g_n(T)$ are unknotted solid tori, the map $g_n$ extends to a homeomorphism of $\mathbb{R}^3$ if and only if $g_n$ takes a longitudinal curve of $T$ to a longitudinal curve of $T'$. If this is not the case, we can first take a twisting homeomorphism of $T$ to itself that preserves the subsets $A, B$, and $V^i$ of $T$ so that the compositions of the twisting homeomorphism and our $g_n$ takes a longitudinal curve of $T$ to a longitudinal curve of $T'$. Thus we may assume that $g_n$ extends to a homeomorphism of $\mathbb{R}^3$ to itself. □

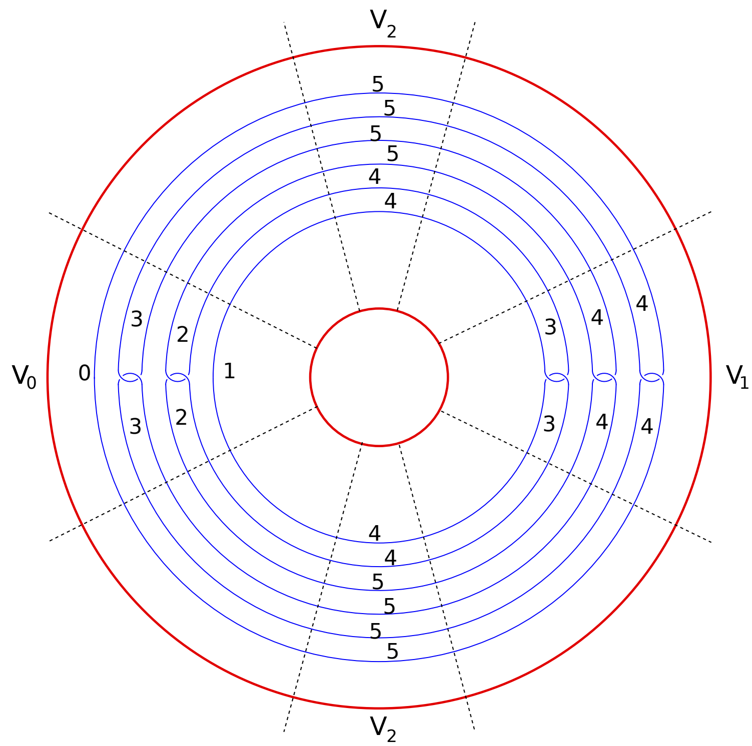



FIGURE 3. Labelled Regions on Tori in Gabai Link

3.3. **Construction.** We will now inductively construct a Gabai 3-manifold corresponding to a sequence $n_1, n_2, n_3, \dots$ of positive integers, with special subsets corresponding to the subsets of $T$ and $T_n'$ just described. Let $T_0 = T$. Let $h_1 : T \to R^3$ be given by $g_{n_1}^{-1}$ and let $T_1 = h_1(T)$. Let $A_1 = h_1(A)$, $B_1 = h_1(B)$, and $V_1^i = h_1(V^i)$. Note that the pair $(T_0, T_1)$ is homeomorphic to $(T_{n_1}', T)$ via $g_{n_1}$and so forms a Gabai Link of index $2n_1$. It follows immediately from Lemma 3.2 and the definitions of the various subsets that:

$$A_1 \cap T_0 = A \text{ and } B_1 \cap T_0 = B \tag{2a}$$

$$V^0 \subset V_1^0 \text{ and for } i > 0, V^i \subset V_1^j, \text{ where } j < i. \tag{2b}$$

Inductively assume that homeomorphisms $h_i : T \to R^3$ have been described for $i \le k$ and that $A_i = h_i(A)$, $B_i = h_i(B)$, and $V_i^j = h_i(V^j)$ for $i \le k$. Also assume that for each $i \le k$:

$$A_i \cap T_{i-1} = A_{i-1} \text{ and } B_i \cap T_{i-1} = B_{i-1} \tag{3a}$$

$$V_{i-1}^0 \subset V_i^0 \text{ and for } j > 0, V_{i-1}^j \subset V_i^\ell, \text{ where } \ell < j \tag{3b}$$

$$\text{the pair } (T_{i-1}, T_i) \text{ is a Gabai Link of index } 2n_i. \tag{3c}$$

For the inductive step, let $h_{k+1} : T \to R^3$ be given by $h_k \circ g_{n_{k+1}}^{-1}$ and let $T_{k+1} = h_{k+1}(T)$, $A_{k+1} = h_{k+1}(A)$, $B_{k+1} = h_{k+1}(B)$, and $V_{k+1}^j = h_{k+1}(V^j)$. Note that the pair $(T_k, T_{k+1})$ is then homeomorphic to $(T'_{n_{k+1}}, T)$ via the homeomorphsm $g_{n_{k+1}} \circ h_k^{-1}$ and so forms a Gabai Link of index $2n_{k+1}$. This shows that Statement (3c) holds when $i = k+1$. Properties (3a) and (3b) for $i = k+1$ follow by applying $h_{k+1}$ to properties (1a) and (1b) from Lemma 3.2. This completes the verification of the inductive step and shows that the following lemma holds.

**Lemma 3.3.** *The Gabai 3-manifold* $G = \bigcup_{m=0}^{\infty} T_m$ *constructed as above satisfies the properties listed in (3a), (3b), and (3c) for all* $i > 0$.

3.4. **Main Result on Gabai Manifolds.** Using the notation from the previous subsection we can state and prove the main result about Gabai manifolds.

**Theorem 3.4.** *Let* $G = \bigcup_{m=0}^{\infty} T_m$ *be a Gabai contractible 3-manifold where each* $T_m$ *is a solid torus and consecutive tori form a Gabai Link. Then* $G$ *satisfies the double 3-space property.*

Proof. The key to the proof is that in the Gabai manifold $G$, we may assume that the conditions in Lemma 3.3 are satisfied.

To show that $G$ satisfies the double 3-space property, we choose the closed sets $A' = \cup_{n=0}^{\infty} A_n$ and $B' = \cup_{n=0}^{\infty} B_n$. Recall that $A_n = h_n(A) = h_n(h(B^2 \times C_1))$ and that $B_n = h_n(B) = h_n(h(B^2 \times C_2))$. We claim that $M = G - A'$, $N = G - B'$ and $M \cap N = G - (A' \cup B')$ are each homeomorphic to $\mathbb{R}^3$.

We first show $M \cap N = G - (A' \cup B')$ is homeomorphic to $\mathbb{R}^3$. It suffices to show that $M \cap N$ is an increasing union of copies of $\mathbb{R}^3$ [Bro61]. First notice that Int $V_n^0 \subset T_n$ is homeomorphic to $\mathbb{R}^3$ since it is the product of an open interval and an open 2-cell. Next notice that $M \cap N = \bigcup_{n=0}^{\infty} \text{Int } V_n^0$ because any point $p$ in $M \cap N$ must lie in the interior of some $V_m^i$ and therefore lies in the interior of $V_{m+i}^0$ by condition (3b) in Lemma 3.3. Again by condition (3b) in Lemma 3.3, the Int $V_n^0$ are nested. So $M \cap N$ is an increasing union of copies of $\mathbb{R}^3$, and so is homeomorphic to $\mathbb{R}^3$.

The proofs that $M$ and $N$ are homeomorphic to $\mathbb{R}^3$ are similar, so we will just focus on $M$. Let $W_0 = V_0 \cup V_1 \cup (B^2 \times [2/3, 1]) \subset T$ and let $W_i = V_{i+1} \cap (B^2 \times [0, 1/3]) \subset T$. Then $T - \bigcup_{i=0}^{\infty} W_i$ is precisely $B^2 \times A$. Let $W_n^i = h_i(W_n)$. Then as in the preceeding paragraph, by the conditions in Lemma 3.3, $M = \bigcup_{n=0}^{\infty} \text{Int } W_n^0$ which is an increasing union of copies of $\mathbb{R}^3$. So $M$ is homeomorphic to $\mathbb{R}^3$. □

**Corollary 3.5.** *There are uncountably many distinct contractible 3-manifolds with the double 3-space property.*

PROOF. This follows directly from Theorem 3.4 and the discussion following Definition 3.1. □

## 4. Interlacing Theory

**Definition 4.1.** Let $A$ and $B$ be finite subsets of a simple closed curve $J$ each containing $k$ points. We say $(A, B)$ is a *$k$-interlacing of points* if each component of $J - A$ contains exactly one point of $B$.

**Definition 4.2.** Let $A$ and $B$ be disjoint compact sets. We say that $(A, B)$ is a *$k$-interlacing for a simple closed curve* $J$ if there exist finite subsets $A' \subset A \cap J$ and $B' \subset B \cap J$ so that $(A', B')$ is a $k$-interlacing of points, but it is impossible to find such subsets that form a $(k+1)$-interlacing of points. If either $A \cap J = \emptyset$ or $B \cap J = \emptyset$, then we say that $(A, B)$ is a 0-interlacing.

**Theorem 4.3** (**Interlacing Theorem for a Simple Closed Curve**)**.** *If $A$ and $B$ are disjoint compact sets and $J$ is a simple closed curve, then $(A, B)$ is a $k$-interlacing for some non-negative integer $k$.*

PROOF. If $A \cap J = \emptyset$ or $B \cap J = \emptyset$, then $(A, B)$ is a 0-interlacing. Otherwise, using compactness, it is possible to cover $A \cap J$ with a finite collection of non-empty, connected, disjoint open sets $U_1, U_2, \ldots, U_m$ and cover $B \cap J$ with a finite collection of non-empty, connected, disjoint open sets $V_1, V_2, \ldots, V_n$ so that the $U_i$ and $V_j$ are also disjoint. If $A' \subset A$ and $B' \subset B$ so that $(A', B')$ is a $k$-interlacing of points for $J$, then $A'$ contains at most one point from each $U_i$ and $B'$ contains at most one point from each $V_j$. So there is a bound on $k$, and our theorem is proved. □

**Theorem 4.4** (**Neighborhood Interlacing Theorem for Simple Closed Curves**)**.** *If $(A, B)$ is a $k$-interlacing for a simple closed curve $J$, then there are open neighborhoods $U$ and $V$ of $A \cap J$ and $B \cap J$, respectively, in $J$ so that if $\overline{A}$ and $\overline{B}$ are disjoint compact sets with $A \cap J \subset \overline{A} \cap J \subset U$ and $B \cap J \subset \overline{B} \cap J \subset V$, then $(\overline{A}, \overline{B})$ is also a $k$-interlacing.*

PROOF. As in the proof of the preceding theorem find the non-empty, connected, disjoint open sets $U_i$ and $V_i$, but in addition we may assume that $m = n = k$. Let $U = \bigcup_{i=1}^{m} U_i$ and $V = \bigcup_{i=1}^{n} V_i$. □

**Theorem 4.5** (**Meridional Disk with Holes Theorem**)**.** [Wri89, Theorem A6] *Let $H$ be a properly embedded disk with holes in a solid torus $T$. Then $H$ is a meridional disk with holes if and only if the inclusion $f : H \to T$ lifts to a map $\hat{f}$ from $H$ to the universal cover $\widetilde{T} = B^2 \times \mathbb{R}$ and $\hat{f}(H)$ separates $\widetilde{H}$ into two unbounded components.*

**Definition 4.6.** Let $A_1, A_2, \ldots, A_k, B_1, B_2, \ldots, B_k$ be disjoint meridional disks with holes in a solid torus $T$. Let $A = \cup_{i=1}^{k} A_i$ and $B = \cup_{i=1}^{k} B_i$. We say that $(A, B)$ is a *$k$-interlacing collection of meridional disks with holes* if each component of $T - A$ contains exactly one $B_i$.

**Definition 4.7.** Let $A$ and $B$ be disjoint compact sets. We say that $(A, B)$ is a *$k$-interlacing for a solid torus $T$* if there exist disjoint meridional disks with holes in $T$, $A_1, A_2, \ldots, A_k, B_1, B_2, \ldots, B_k$ with $A' = \cup_{i=1}^k A_i \subset A$ and $B' = \cup_{i=1}^k B_i \subset B$ so that $(A', B')$ is a $k$-interlacing collection of meridional disks with holes, but it is impossible to find such subsets that form a $(k+1)$-interlacing collection of meridional disks with holes. If either $A$ or $B$ fails to contain a meridional disk with holes in $T$, then we say that $(A, B)$ is a 0-interlacing.

**Lemma 4.8.** *If $(A, B)$ is a $k$-interlacing collection of meridional disks with holes for the solid torus $T$ and $J$ is a simple closed curve core for $T$, then $(A, B)$ is an $n$-interlacing of $J$ where $n \geq k$.*

Proof. If $k = 0$ or $k = 1$ the proof is quite easy. Each component of $T - A$ contains exactly one meridional disk with holes component of $B$. Let $J$ be a simple closed curve core for $T$. Since each disk with holes component of $A$ is interior essential, $J$ must meet each component of $A$. Let $U$ be a component of $J - A$ so that the endpoints of the closure of $U$ are in different components of $A$. Then $U$ must meet a component of $B$ since each component of $B$ is interior essential. Since there are at least $k$ such components of $J - A$ with endpoints of the closure in different components of $A$, $(A, B)$ must be at least a $k$-interlacing for $J$. Thus we see that $(A, B)$ is an $n$-interlacing of $J$ where $n \geq k$. □

**Theorem 4.9** (**Interlacing Theorem for a Solid Torus**). *If $A$ and $B$ are disjoint compact sets and $T$ is a solid torus, then $(A, B)$ is a $k$-interlacing of $T$ for some non-negative integer $k$.*

Proof. We just need to show that the interlacing number of $(A, B)$ with respect to $T$ is bounded. Let $J$ be a core simple closed curve for the solid torus. The interlacing number of $(A, B)$ with respect to $T$ is less than or equal to the interlacing number of $(A, B)$ with respect to $J$ which is well-defined by the Interlacing Theorem for simple closed curves. □

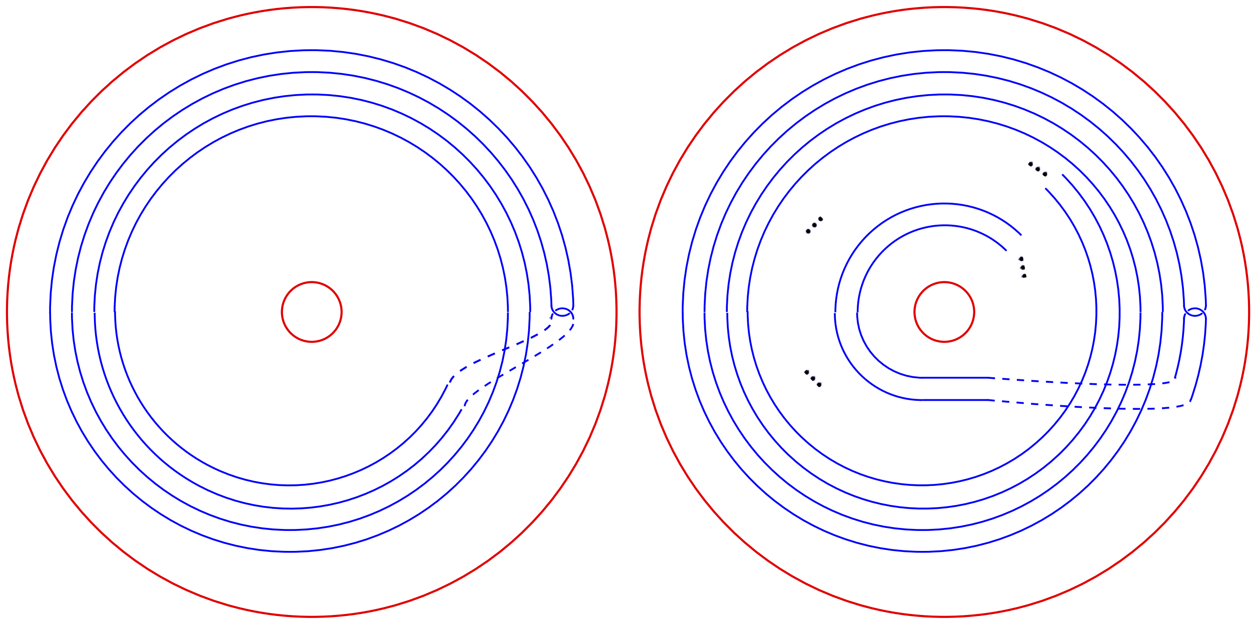
## 5. McMillan Contractible 3-Manifolds Do Not Satisfy the Double 3-space Property

There is an alternative generalization of a Whitehead Link that was used by McMillan [McM62] to show the existence of uncountably many contractible 3-manifolds. We call these links *McMillan Links*.

**Definition 5.1.** Let $n$ be a positive integer. A *McMillan Link* of geometric index $2n$ is a pair of solid tori $T' \subset T$ so that $T'$ is embedded in $T$ as illustrated in Figure 4 for a McMillan Link of index 4 and of index $2n$.

**Definition 5.2.** If $M$ is a genus one 3-manifold with defining sequence $(T_i)$, then we say that $M$ is a *McMillan Contractible 3-manifold* if for each $i$, $T_i \subset T_{i+1}$ is a McMillan Link of geometric index at least 4.

There are immediate results that follow from the previous section.



(a) McMillan Link: Index 4 (b) McMillan Link: Index 2n

FIGURE 4. McMillan Links

**Theorem 5.3** (**Interlacing Theorem for a McMillan Link**). *Suppose that $A$ and $B$ are disjoint planar 2-manifolds properly embedded in a solid torus $T$ so that $(A, B)$ is a $k$-interlacing for $T$. If $T'$ is a McMillan Link of geometric index $2n$ in $T$ so that $T'$ is in general position with respect to $A \cup B$, then $(A, B)$ is an $m$-interlacing for $T'$ where $m \geq 2nk - 1$.*

PROOF. Let $p : \widetilde{T} \to T$ be the projection map from the $n$-fold cover of $T$. Since $(A, B)$ is a $k$-interlacing for $T$, there exist disjoint meridional disks with holes, $A_1, A_2, \ldots, A_k$ and $B_1, B_2, \ldots, B_k$ with $A' = \cup_{i=1}^{k} A_i \subset A$ and $B' = \cup_{i=1}^{k} B_i \subset B$ so that $(A', B')$ is a $k$-interlacing collection of meridional disks with holes for $T$. Set $\widetilde{A}' = p^{-1}(A')$ and $\widetilde{B}' = p^{-1}(B')$. Using the Meridional Disk with Holes Theorem, we see that $(\widetilde{A}', \widetilde{B}')$ is an $nk$-interlacing collection of meridional disks with holes for $\widetilde{T}$. Let $\widetilde{i} : T' \to \widetilde{T}$ be a lift of the inclusion map $i : T' \to T$. Then $T'' = \widetilde{i}(T')$ is a Whitehead Link in $\widetilde{T}$. By [Wri89, Lemma A10] $(\widetilde{A}', \widetilde{B}')$ is an $m$-interlacing of $T''$ where $m \geq 2nk - 1$. It now follows that $(A, B)$ is an $m$-interlacing for $T'$ for $m \geq 2nk - 1$. □

**Corollary 5.4.** *In the previous theorem, if $T'$ has geometric index at least 4, then $m > k$.*

We now prove some lemmas that are needed in proving that McMillan contractible 3-manifolds do not have the double 3-space property.

**Lemma 5.5.** *Let $H$ be a properly embedded 2-manifold in a solid torus $T$ so that each component of $H$ is an interior-inessential disk with holes. Then there is an essential simple closed curve in $T$ that misses $H$.*

PROOF. Let $J$ be an oriented essential simple closed curve in $T$ that is in general position with respect to $H$. The proof is by induction on the number of points in $J \cap H$. Consider a component $H'$ of $H$ that meets $J$. Choose an orientation on $H'$. Since $H'$ is interior-inessential, the algebraic intersection number of $J$ and $H'$ is zero (meaning that there are the same number of positive and negative intersections). Let $p, q \in J \cap H'$ be points with opposite orientations. The points $p$ and $q$ separate $J$ into two components $J_1$ and $J_2$. Let $A$ be an arc in $H'$ between $p$ and $q$ that misses all other points of $J \cap H'$. Then $J_1 \cup A$ and $J_2 \cup A$ are simple closed curves. If $J_1 \cup A$ and $J_2 \cup A$ are both inessential in $T$, then so is $J$, so at least one of $J_1 \cup A$ and $J_2 \cup A$ is essential in $J$. We suppose that $J' = J_1 \cup A$ is essential in $T$. Using a collar on $H'$, we can push $J_1$ off $H$ to get an essential simple closed curve $J''$ that meets $H$ in two fewer points than $J$. □

**Lemma 5.6.** *Let $M$ be a 3-manifold so that $M = U \cup V$ where $U$, $V$ are homeomorphic to $\mathbb{R}^3$. Let $T \subset M$ be a solid torus so that for every essential simple closed curve $J \subset T$, $J \not\subset U$ and $J \not\subset V$. Let $C = M - U$ and $D = M - V$. Then any neighborhood of $T \cap C$ in $T$ contains a meridional disk with holes.*

Proof. Notice that by DeMorgan's Law, $C \cap D = \emptyset$. Since $T \not\subset U$ and $T \not\subset V$, then $C' = T \cap C \neq \emptyset$ and $D' = T \cap D \neq \emptyset$. So $C'$ and $D'$ are disjoint non-empty compact subsets of $T$. Let $N$ be an open neighborhood of $C'$ in $T$ that misses $D'$. Let $K = T - N$. Then $K$ is a compact set in $U$ that contains $D'$. Since $U$ is homeomorphic to $\mathbb{R}^3$, $K$ is contained in the interior of a 3-ball $B \subset U$ with boundary a 2-sphere $S$ that we may suppose is in general position with respect to $T$. Notice that $C'$ and $D'$ are in separate components of $M - S$ and so $S \cap T = \emptyset$ is impossible. Also $S \subset \text{Int } T$ is impossible because this would allow for an essential simple closed curve in $T$ that would lie in either $U$ or $V$. Thus the set $H = S \cap T \neq \emptyset$ lies in the neighborhood $N$ of $C'$, and each component of $H$ is a disk with holes. If each component is interior-inessential, then, by the previous lemma, there is an essential simple closed curve $J$ in $T$ that misses $H$. So $J$ lies in a component of $M - S$ and must miss either $C$ or $D$. So $J \subset U$ or $J \subset V$ which is a contradiction. Thus at least one of the components of $H$ must be interior-essential and thus a meridional disk with holes.

□

**Theorem 5.7.** *No McMillan contractible 3-manifold $M$ can be expressed as the union of two copies of $\mathbb{R}^3$.*

Proof. Let $T_i$ be a defining sequence for $M$ so that $M = \cup_{i=0}^{\infty} T_i$ . Suppose $M = U \cup V$ where $U$, $V$ are homeomorphic to $\mathbb{R}^3$. Then by Lemma 2.6, for each essential simple closed curve $J' \subset T_i$, $J' \not\subset U$ and $J' \not\subset V$. Let $C = M - U$ and $D = M - V$. Then by Lemma 5.6 each neighborhood of $T_i \cap C$ and each neighborhood of $T_i \cap D$ contains a meridional disk with holes for $T_i$.

Let $J$ be a simple closed curve core of $T_0$. Let $n$ be the interlacing number of $(J \cap C, J \cap D)$. Let $\overline{C}$ and $\overline{D}$ be closed neighborhoods in $J$ of $J \cap C$ and $J \cap D$, respectively so that the interlacing number for $(\overline{C}, \overline{D})$ is also $n$. Let $H_C$ be a meridional disk with holes in a neighborhood of $C \cap T_n$ and $H_D$ be a meridional disk with holes in a neighborhood of $D \cap T_n$ so that

(1) $H_C \cap H_D = \emptyset$
(2) $H_C \cap J \subset \overline{C}$, $H_D \cap J \subset \overline{D}$
(3) $H_C$ an $H_D$ are in general position with respect to $T_i, 0 \leq i \leq n$.

By Corollary 5.4 the interlacing number of $(H_C \cap T_0, H_D \cap T_0)$ in $T_0$ is greater than $n$. This implies that the interlacing number of $(\overline{A}, \overline{B})$ in $J$ is also greater than $n$, a contradiction to Lemma 4.8 □

**Corollary 5.8.** *There are uncountably many distinct contractible 3-manifolds that fail to have the double 3-space property.*

Proof. This follows directly from Theorem 5.7 and the discussion following Definition 3.1. □

## 6. Questions and Acknowledgments

The results in this paper produce two infinite classes of genus one contractible 3-manifolds, one of which has the double 3-space property and one of which does not. There are many genus one contractible 3-manifolds that do not fit into either of these two classes. This leads to a number of questions.

**Question 6.1.** *Is it possible to characterize which genus one contractible 3-manifolds have the double 3-space property?*

**Question 6.2.** *Is it possible to characterize which contractible 3-manifolds have the double 3-space property?*

**Question 6.3.** *Is there a contractible 3-manifold M which is the union of two copies of* $\mathbb{R}^3$*, but which does not have the double 3-space property?*

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