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Unknotting Numbers and Minimal Knot Diagrams

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The unknotting number of a knot is the minimum number of crossing changes necessary to obtain the unknot, taken over all regular projections of the knot. For a long time, it was an open question whether one could restrict investigation of crossing changes to minimal projections and still obtain the correct unknotting number. This question was answered negatively by Bleiler in [1] and Nakanishi in [5], both giving the counterexample of the knot 5 1 4 (in Conway's notation). The minimal projection of this knot requires three crossing changes to become unknotted, while there is a nonminimal projection which requires only two. We shall generalize the procedure which is used to show that 5 1 4 attains its unknotting number only on a nonminimal projection in order to exhibit an infinite number of knots with this property.

To this end, we denote the knot given by $c_1\ c_2\ \ldots\ c_j$ in Conway's notation by $C(c_1, c_2, \ldots, c_j)$ and consider the knots $C(2k+1, 1, 2k)$, where $k \geq 2$ is an integer. We shall demonstrate that, while the minimal projections of these knots (see Figure 1 below) cannot be unknotted in k crossing changes, there are nonminimal projections (as in Figure 2 below) which can be.

At this point, we shall assume that we are making all our crossing changes simultaneously (so that the order of changing is irrelevant). The effect of not assuming this will be examined later and will lead to an interesting question.

By a result of Kauffman, Thistlethwaite, and Murasugi (see [3]), the projection $C(2k+1, 1, 2k)$ is minimal because it is reduced alternating. Furthermore, Tait's flyping conjecture (see [4]) tells us this minimal projection is unique up to flyping, which will not change the number of crossing changes necessary to unknot a particular diagram. To see that this projection cannot be unknotted in k or fewer crossing changes, we use induction.

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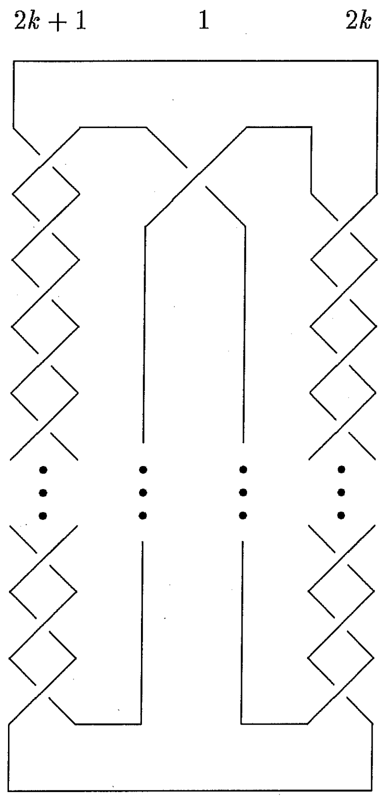

Figure 1

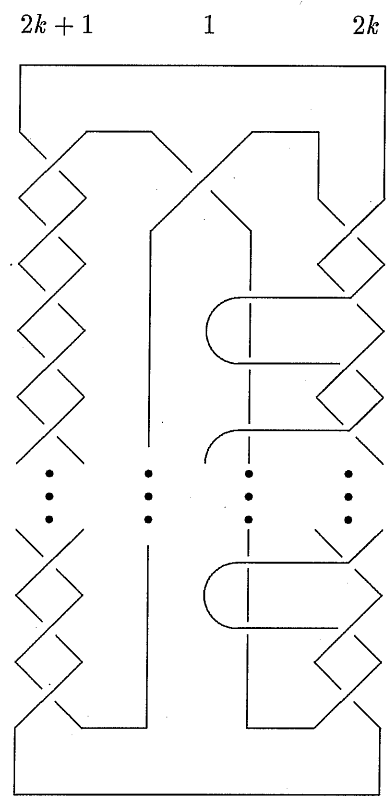

Figure 2

The case where $k = 2$ is verified by Bleiler in [1], so we proceed to show that if it holds for k, it holds for $k + 1$. Hence we examine the effect of at most $k + 1$ crossing changes on the knot $C(2k + 3, 1, 2k + 2)$. We split our task into five cases.

Case (i) If our set of crossing changes includes at least one from both the $2k + 3$ and $2k + 2$ portions of the knot, then we cannot possibly have unknotted our knot. For after making our (at most) $k + 1$ changes, we examine our diagram before allowing any ambient isotopies. Since at most k crossing changes could have occured in either of the $2k + 3$ of the $2k + 1$ section, it follows that in both sections, there is some crossing which was changed but whose nearest crossing above or below was not. In this case, we can untwist these two crossings while leaving the rest of the crossings unaffected (the case where the crossing below is unchanged is shown in Figure 3 below). After doing this in both the $2k + 3$ and the $2k + 1$ parts of the knot, we are left with the minimal projection of $C(2k + 1, 1, 2k)$, modified at (at most) $k - 1$ crossings, which we know by our inductive hypothesis is knotted.

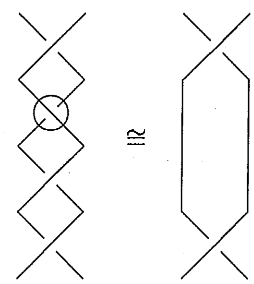

Figure 3

Case (ii) If we change n crossings in the $2k+2$ side, where $0 \leq n \leq k+1$, we get the knot $C(2k+3, 1, 2k+2-2n)$. Except when $n = k+1$, this is a reduced alternating knot, and is thus a minimal projection. Since it has $(4k+6-2n) > 2$ crossings and is minimal, it is not the unknot. If $n = k+1$, then we are left with the knot $C(2k+3, 1, 0)$, whose minimal projection is given by $C(2k+3)$. This is just the $(2k+3, 2)$-torus knot, and not the unknot.

Case (iii) If we change n crossings in the $2k+3$ side, where again $0 \leq n \leq k+1$, we get the knot $C(2k+3-2n, 1, 2k+2)$ which is knotted for the same reason as in Case (ii).

Case (iv) Changing the "1" crossing and then n crossings in the $2k+2$ section of the knot, where this time $0 \leq n \leq k$, we obtain the knot $C(2k+3, -1, 2k+2-2n)$. Here we use the identity

$$x + (-1 + (x+1)^{-1})^{-1} = (x-2) + (1 + (x-1)^{-1})^{-1} = (x^2 - x - 1)/x.$$

This tells us that the knot $C(2k+3, -1, 2k+2-2n)$ is equivalent to the knot $C(2k+1, 1, 2k-2n)$ since their associated continued fractions are equal (see [2]). But for the same reasons as in Case (ii), this is knotted, so our $k+1$ crossings are not sufficient to unknot our original knot.

Case (v) Changing the "1" crossing and then n crossings in the $2k+3$ section of the knot, with $0 \leq n \leq k$ again, we get the knot $C(2k+3-2n, -1, 2k+2)$. As in Case (iv), we see that this is equivalent to $C(2k+1-2n, 1, 2k)$, which is knotted as in Case (iii). Thus $k+1$ crossings will not unknot our original knot in this case.

Therefore, in all cases, we cannot unknot the projection $C(2k+3, 1, 2k+2)$ in $k+1$ crossing changes. By induction, then, we cannot unknot the projection $C(2k+1, 1, 2k)$ in k crossing changes, for all $k \geq 2$.

Now we show that the nonminimal projection pictured in Figure 2 above can be unknotted in k crossing changes. After changing the "1" crossing and the $k-1$ crossings in the loops around the center strand in our nonminimal projection, we get the unknot. To see this, note that (as shown in Figure 4 below) two twists each from the $2k+1$ and the $2k$ parts of the knot can be unraveled until all our added loops have been used and we are left with the knot $C(3,-1,2)$, which is the unknot. Hence this projection can be unknotted in k crossing changes.

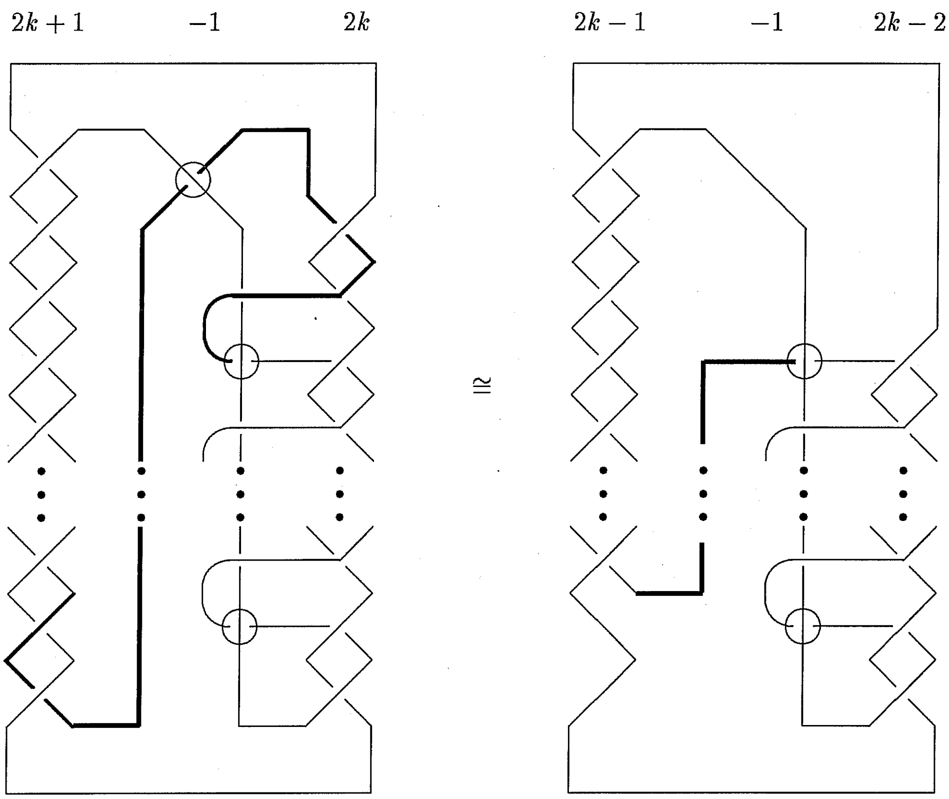

Figure 4

We have, then, an infinite number of knots $C(2k+1,1,2k)$, with $k \geq 2$, whose unknotting number cannot be realized by considering only minimal projections.

Also, it should be noted that the class of knots obtained by changing all the circled crossings in the nonminimal projection of $C(2k+1,1,2k)$ with the exception of the "1" crossing shows that there exist knots of arbitrarily large crossing number with unknotting number one. We realize this since, upon examining any of our loops created by a Reidemeister move and a crossing change, we see that adding this loop does not change the crossing number of the original knot. This is clear from Figure 5 below, in which the projection on the right is minimal (since it is reduced alternating) and has the same number of crossings as the minimal projection of the knot $C(2k+1,1,2k)$.

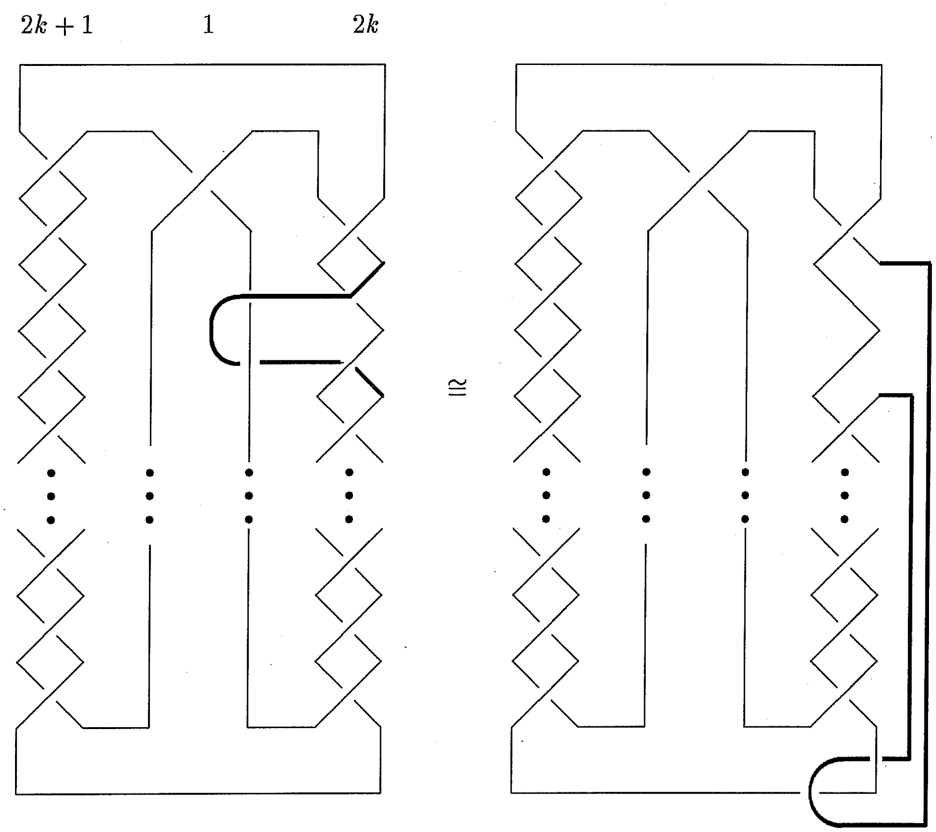

Figure 5

If, on the other hand, we relax our requirement that all the crossing changes be done simultaneously and allow ourselves to effect ambient isotopies between crossing changes, we notice an interesting result. In this case, when we examine the knots $C(2k+1,1,2k)$ for $k \geq 2$, we see that we can, in some sense, restrict ourselves to minimal diagrams when computing the unknotting number. This is because, first of all, changing the "1" crossing yields the knot $C(2k+1,-1,2k)$. We saw in Case (iv) above that this can be changed to $C(2k-1,1,2k-2)$ by an ambient isotopy. From here, we can alternate "1" crossing changes and ambient isotopies until we reach the knot $C(3,1,2)$ after $k-1$ crossing changes. From there, changing the "1" crossing results in the unknot. Note that each projection on which a crossing change is made is minimal since it is reduced alternating. Thus, by allowing ourselves to effect an ambient isotopy to bring ourselves to a minimal projection before each crossing change, we can in fact unknot these examples in the same number of changes as was exhibited in the nonminimal projection with simultaneous crossing changes. For the particular case of $C(5,1,4)$, this procedure of alternating between crossing changes and ambient isotopies actually achieves the unknotting number, which is known to be two. Thus the following question arises:

Can the unknotting number always be realized by a sequence of crossing changes, each of which takes place on a minimal projection?

The author conjectures an affirmative answer to this question; if this is the case, a simple method for calculating the unknotting number of a knot K is provided. One would need to consider all knots obtainable by a single crossing change of a minimal diagram of K, and, of these knots, find the minimum unknotting number. Adding one to this minimum would then yield the unknotting number of K.

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