MTH 420-520 Eigenvalue/eigenvector computations useful in dynamics.
EXTRA CREDIT

Assume that a matrix $A \in \mathbb{R}^{n\times n}$ is symmetric and positive definite. The goal of this assignment is to understand why the dynamics of an initial value problem for

$$\frac{dx}{dt} + Ax = 0 \tag{1}$$

is dramatically different from the dynamics of

$$\frac{d^2x}{dt^2} + Ax = 0 \tag{2}$$

The results of this assignment will be used later when examining the solutions to the heat/diffusion equation $u_t - u_{xx} = 0$ (which resemble (1) and are *dissipative*) and to the wave equation $u_{tt} - u_{xx} = 0$ (which resemble (2) and are *conservative*).

Part A. Assume $n = 1, x \in R$, and $A > 0$ is a constant.

Write out the general solution to (1) and to (2).

$$\text{Solution to (1) } x(t) =$$
$$\text{Solution to (2) } x(t) =$$

Part B. Assume $n > 1, x \in R^n$, and $A \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix. Such a matrix is diagonalizable, that is, there is a nonsingular matrix (of eigenvectors) V and a diagonal matrix Λ with the real positive eigenvealues $0 < \lambda_1 \leq \lambda_2 \leq \ldots \lambda_n$ being the diagonal elements of Λ, so that we can decompose

$$A = V\Lambda V^{-1} \tag{3}$$

(In fact, one can prove that for an spd matrix A, the eigenvectors are orthogonal, but this will not be needed here).

To solve (1), we rewrite it (after multiplying through by V^{-1} as

$$\frac{dV^{-1}x}{dt} = \Lambda(V^{-1}x).$$

Now we change variable $w = V^{-1}x$ and solve the system $\frac{dw}{dt} = \Lambda w$ rewritten as

$$\frac{dw_j}{dt} = \lambda_j w_j.$$

Complete: the general solution $w_j(t)$ is,

$$w_j(t) =$$

After you change the basis from w back to x, the general solution vector $x(t)$ is

$$x(t) =$$

Part C. Now we rewite (2) as follows. We introduce a new variable $y = \frac{dx}{dt} = x'$ and rewrite the equation (2) as

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} & I \\ -A & \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = B\begin{bmatrix} x \\ y \end{bmatrix}, \tag{4}$$

where the blank space denotes zeros, I denotes the identity matrix of appropriate size, and where B denotes the $2n \times 2n$ matrix that arose on the right hand side.

Complete: Assuming that the eigenvalues of A are the positive real numbers $0 < \lambda_1 \leq \lambda_2 \leq \ldots \lambda_n$ (as in part B), show that each eigenvalue μ of B is given as $\mu = \pm i\sqrt{\lambda_j}$ for some $1 \leq j \leq n$.