[Lay, Lin. Alg. & Apps.]

SOLUTION TO PRACTICE PROBLEM

Examine the Casorati matrix:

$$C(k) = \begin{bmatrix} 2^k & 3^k \sin\frac{k\pi}{2} & 3^k \cos\frac{k\pi}{2} \\ 2^{k+1} & 3^{k+1} \sin\frac{(k+1)\pi}{2} & 3^{k+1} \cos\frac{(k+1)\pi}{2} \\ 2^{k+2} & 3^{k+2} \sin\frac{(k+2)\pi}{2} & 3^{k+2} \cos\frac{(k+2)\pi}{2} \end{bmatrix}$$

Set $k = 0$ and row reduce the matrix to verify that it has three pivot positions and hence is invertible:

$$C(0) = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & 0 \\ 4 & 0 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & -2 \\ 0 & 0 & -13 \end{bmatrix}$$

The Casorati matrix is invertible at $k = 0$, so the signals are linearly independent. Since there are three signals, and the solution space H of the difference equation has dimension 3 (Theorem 17), the signals form a basis for H, by the Basis Theorem.

4.9 Applications to Markov Chains

The Markov chains described in this section are used as mathematical models of a wide variety of situations in biology, business, chemistry, engineering, physics, and elsewhere. In each case, the model is used to describe an experiment or measurement that is performed many times in the same way, where the outcome of each trial of the experiment falls into one of several specified possible outcomes, and where the outcome of one trial depends only on the immediately preceding trial.

For example, if the population of a city and its suburbs were measured each year, then a vector such as

$$\mathbf{x}_0 = \begin{bmatrix} .60 \\ .40 \end{bmatrix} \qquad (1)$$

could indicate that 60% of the population lives in the city and 40% in the suburbs. The decimals in $\mathbf{x}_0$ add up to 1 because they account for the entire population of the region. Percentages are more convenient for our purposes here than population totals.

A vector with nonnegative entries that add up to one is called a **probability vector.** A **stochastic matrix** is a square matrix whose columns are probability vectors. A **Markov chain** is a sequence of probability vectors $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots$, together with a stochastic matrix P, such that

$$\mathbf{x}_1 = P\mathbf{x}_0, \quad \mathbf{x}_2 = P\mathbf{x}_1, \quad \mathbf{x}_3 = P\mathbf{x}_2, \quad \ldots$$

Thus the Markov chain is described by the first-order difference equation

$$\mathbf{x}_{k+1} = P\mathbf{x}_k \quad \text{for } k = 0, 1, 2, \ldots$$

When a Markov chain of vectors in $\mathbb{R}^n$ describes a system or a sequence of experiments, the entries in $\mathbf{x}_k$ list, respectively, the probabilities that the system is in each

of n possible states, or the probabilities that the outcome of the experiment is one of n possible outcomes. For this reason, $\mathbf{x}_k$ is often called a **state vector.**

EXAMPLE 1 In Section 1.9 we examined a model for population movement between a city and its suburbs. See Fig. 1. The annual migration between these two parts of the metropolitan region was governed by the ***migration matrix*** M:

$$\begin{array}{c} \text{From:} \\ \begin{array}{cc} \text{City} & \text{Suburbs} \end{array} \end{array} \quad \text{To:}$$

$$M = \begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix} \begin{array}{l} \text{City} \\ \text{Suburbs} \end{array}$$

That is, each year 5% of the city population moves to the suburbs, and 3% of the suburban population moves to the city. The columns of M are probability vectors, so M is a stochastic matrix. Suppose that the 1990 population of the region is 600,000 in the city and 400,000 in the suburbs. Then the initial distribution of the population in the region is given by $\mathbf{x}_0$ in (1) above. What is the distribution of the population in 1991? in 1992?

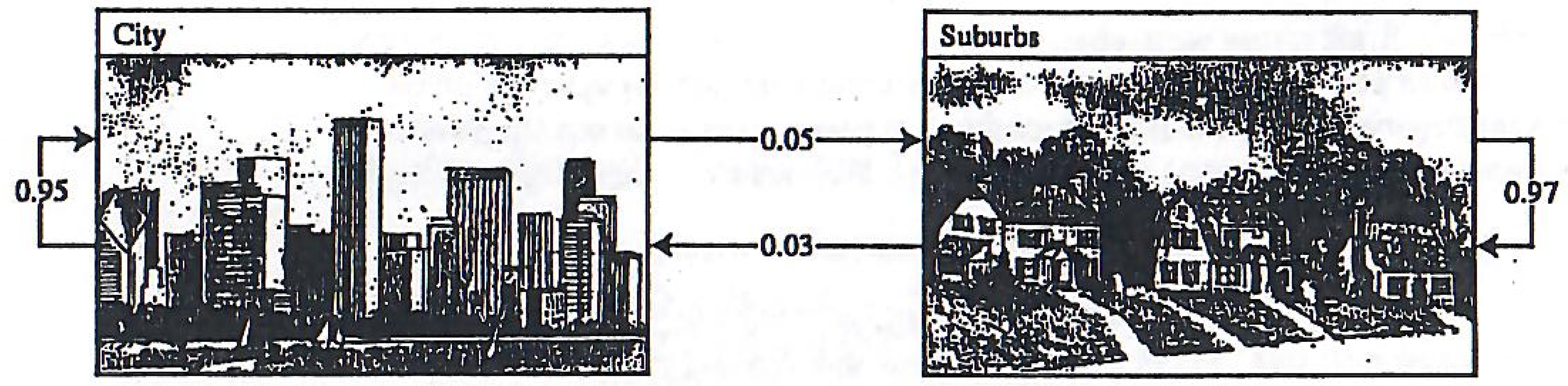

FIGURE 1 Annual percentage migration between city and suburbs.

Solution In Example 2 of Section 1.9, we saw that after one year, the population vector $\begin{bmatrix} 600{,}000 \\ 400{,}000 \end{bmatrix}$ changed to

$$\begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix} \begin{bmatrix} 600{,}000 \\ 400{,}000 \end{bmatrix} = \begin{bmatrix} 582{,}000 \\ 418{,}000 \end{bmatrix}$$

If we divide both sides of this equation by the total population of 1 million, and use the fact that $kM\mathbf{x} = M(k\mathbf{x})$, we find that

$$\begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix} \begin{bmatrix} .600 \\ .400 \end{bmatrix} = \begin{bmatrix} .582 \\ .418 \end{bmatrix}.$$

The vector $\mathbf{x}_1 = \begin{bmatrix} .582 \\ .418 \end{bmatrix}$ gives the population distribution in 1991. That is, 58.2% of the region lived in the city and 41.8% lived in the suburbs. Similarly, the population distribution in 1992 is described by a vector $\mathbf{x}_2$, where

$$\mathbf{x}_2 = M\mathbf{x}_1 = \begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix}\begin{bmatrix} .582 \\ .418 \end{bmatrix} = \begin{bmatrix} .565 \\ .435 \end{bmatrix}$$

EXAMPLE 2 Suppose that the voting results of a congressional election at a certain voting precinct are represented by a vector $\mathbf{x}$ in $\mathbb{R}^3$:

$$\mathbf{x} = \begin{bmatrix} \text{\% voting Democratic (D)} \\ \text{\% voting Republican (R)} \\ \text{\% voting Libertarian (L)} \end{bmatrix}$$

Suppose that we record the outcome of the congressional election every two years by a vector of this type and that the outcome of one election depends only on the results of the preceding election. Then the sequence of vectors that describe the votes every two years may be a Markov chain. As an example of a stochastic matrix P for this chain, we take

$$\begin{array}{c} \text{From:} \\ \begin{array}{cccc} \text{D} & \text{R} & \text{L} & \text{To:} \end{array} \end{array}$$

$$P = \begin{bmatrix} .70 & .10 & .30 \\ .20 & .80 & .30 \\ .10 & .10 & .40 \end{bmatrix} \begin{matrix} \text{D} \\ \text{R} \\ \text{L} \end{matrix}$$

The entries in the first column labeled "D" describe what the persons voting Democratic in one election will do in the next election. Here we have supposed that 70% will vote "D" again in the next election, 20% will vote "R," and 10% will vote "L." A similar interpretation holds for the other columns of P. A diagram for this matrix is shown in Fig. 2.

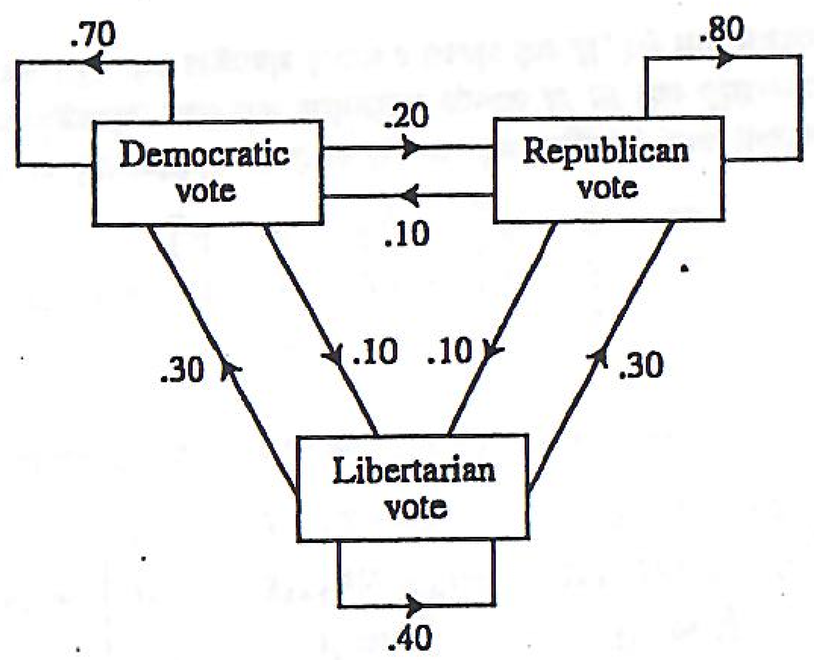

FIGURE 2 Voting changes from one election to the next.

If the "transition" percentages remain constant over many years from one election to the next, then the sequence of vectors that give the voting outcomes forms a Markov chain. Suppose that in one election, the outcome is given by

$$\mathbf{x}_0 = \begin{bmatrix} .55 \\ .40 \\ .05 \end{bmatrix}$$

Determine the likely outcome of the next election and the likely outcome of the election after that.

Solution The outcome of the next election is described by the state vector $\mathbf{x}_1$ and the election after that by $\mathbf{x}_2$, where

$$\mathbf{x}_1 = P\mathbf{x}_0 = \begin{bmatrix} .70 & .10 & .30 \\ .20 & .80 & .30 \\ .10 & .10 & .40 \end{bmatrix}\begin{bmatrix} .55 \\ .40 \\ .05 \end{bmatrix} = \begin{bmatrix} .440 \\ .445 \\ .115 \end{bmatrix}, \quad \begin{matrix} \text{44\% will vote D.} \\ \text{44.5\% will vote R.} \\ \text{11.5\% will vote L.} \end{matrix}$$

$$\mathbf{x}_2 = P\mathbf{x}_1 = \begin{bmatrix} .70 & .10 & .30 \\ .20 & .80 & .30 \\ .10 & .10 & .40 \end{bmatrix}\begin{bmatrix} .440 \\ .445 \\ .115 \end{bmatrix} = \begin{bmatrix} .3870 \\ .4785 \\ .1345 \end{bmatrix}, \quad \begin{matrix} \text{38.7\% will vote D.} \\ \text{47.8\% will vote R.} \\ \text{13.5\% will vote L.} \end{matrix}$$

To understand why $\mathbf{x}_1$ does indeed give the outcome of the next election, suppose that 1000 persons voted in the "first" election, with 550 voting "D," 400 voting "R," and 50 voting "L." (See the percentages in $\mathbf{x}_0$.) In the next election, 70% of the 550 will vote "D" again, 10% of the 400 will switch from "R" to "D," and 30% of the 50 will switch from "L" to "D." Thus the total "D" vote will be

$$.70(550) + .10(400) + .30(50) = 385 + 40 + 15 = 440 \qquad (2)$$

Thus 44% of the vote next time will be for the "D" candidate. The calculation in (2) is essentially the same as that used to compute the first entry in $\mathbf{x}_1$. Analogous calculations could be made for the other entries in $\mathbf{x}_1$, for the entries in $\mathbf{x}_2$, and so on.

Predicting the Distant Future

The most interesting aspect of Markov chains is the study of a chain's long-term behavior. For instance, what can be said in Example 2 about the voting after many elections have passed (assuming that the given stochastic matrix continues to describe the transition percentages from one election to the next)? Or, what happens to the population distribution in Example 1 "in the long run"? Before answering these questions, we turn to a numerical example.

EXAMPLE 3 Let $P = \begin{bmatrix} .5 & .2 & .3 \\ .3 & .8 & .3 \\ .2 & .0 & .4 \end{bmatrix}$ and $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Consider a system whose state is described by the Markov chain $\mathbf{x}_{k+1} = P\mathbf{x}_k$, for $k = 0, 1, \ldots$. What happens to the system as time passes? Compute the state vectors $\mathbf{x}_1, \ldots, \mathbf{x}_{15}$ to find out.

Solution The entries in the vectors that follow have been rounded to four or five significant figures.

$$\mathbf{x}_1 = P\mathbf{x}_0 = \begin{bmatrix} .5 & .2 & .3 \\ .3 & .8 & .3 \\ .2 & 0 & .4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} .5 \\ .3 \\ .2 \end{bmatrix}$$

$$\mathbf{x}_2 = P\mathbf{x}_1 = \begin{bmatrix} .5 & .2 & .3 \\ .3 & .8 & .3 \\ .2 & 0 & .4 \end{bmatrix}\begin{bmatrix} .5 \\ .3 \\ .2 \end{bmatrix} = \begin{bmatrix} .37 \\ .45 \\ .18 \end{bmatrix}$$

$$\mathbf{x}_3 = P\mathbf{x}_2 = \begin{bmatrix} .5 & .2 & .3 \\ .3 & .8 & .3 \\ .2 & 0 & .4 \end{bmatrix}\begin{bmatrix} .37 \\ .45 \\ .18 \end{bmatrix} = \begin{bmatrix} .329 \\ .525 \\ .146 \end{bmatrix}$$

Continuing in this manner, we have

$$\mathbf{x}_4 = \begin{bmatrix} .3133 \\ .5625 \\ .1242 \end{bmatrix}, \quad \mathbf{x}_5 = \begin{bmatrix} .3064 \\ .5813 \\ .1123 \end{bmatrix}, \quad \mathbf{x}_6 = \begin{bmatrix} .3032 \\ .5906 \\ .1062 \end{bmatrix}, \quad \mathbf{x}_7 = \begin{bmatrix} .3016 \\ .5953 \\ .1031 \end{bmatrix}$$

$$\mathbf{x}_8 = \begin{bmatrix} .3008 \\ .5977 \\ .1016 \end{bmatrix}, \quad \mathbf{x}_9 = \begin{bmatrix} .3004 \\ .5988 \\ .1008 \end{bmatrix}, \quad \mathbf{x}_{10} = \begin{bmatrix} .3002 \\ .5994 \\ .1004 \end{bmatrix}, \quad \mathbf{x}_{11} = \begin{bmatrix} .3001 \\ .5997 \\ .1002 \end{bmatrix}$$

$$\mathbf{x}_{12} = \begin{bmatrix} .30005 \\ .59985 \\ .10010 \end{bmatrix}, \quad \mathbf{x}_{13} = \begin{bmatrix} .30002 \\ .59993 \\ .10005 \end{bmatrix}, \quad \mathbf{x}_{14} = \begin{bmatrix} .30001 \\ .59996 \\ .10002 \end{bmatrix}, \quad \mathbf{x}_{15} = \begin{bmatrix} .30001 \\ .59998 \\ .10001 \end{bmatrix}$$

These vectors seem to be approaching $\mathbf{q} = \begin{bmatrix} .3 \\ .6 \\ .1 \end{bmatrix}$. The probabilities are hardly changing from one value of k to the next. Observe that the following calculation is exact (with no rounding error):

$$P\mathbf{q} = \begin{bmatrix} .5 & .2 & .3 \\ .3 & .8 & .3 \\ .2 & 0 & .4 \end{bmatrix}\begin{bmatrix} .3 \\ .6 \\ .1 \end{bmatrix} = \begin{bmatrix} .15 + .12 + .03 \\ .09 + .48 + .03 \\ .06 + 0 + .04 \end{bmatrix} = \begin{bmatrix} .30 \\ .60 \\ .10 \end{bmatrix} = \mathbf{q}$$

When the system is in state $\mathbf{q}$, there is no change in the system from one measurement to the next. ∎

Steady-State Vectors

If P is a stochastic matrix, then a **steady-state vector** (or **equilibrium vector**) for P is a probability vector $\mathbf{q}$ such that

$$P\mathbf{q} = \mathbf{q}$$

It can be shown that every stochastic matrix has a steady-state vector. In Example 3 above, $\mathbf{q}$ is a steady-state vector for P.

EXAMPLE 4 The probability vector $\mathbf{q} = \begin{bmatrix} .375 \\ .625 \end{bmatrix}$ is a steady-state vector for the population migration matrix M in Example 1, because

$$M\mathbf{q} = \begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix}\begin{bmatrix} .375 \\ .625 \end{bmatrix} = \begin{bmatrix} .35625 + .01875 \\ .01875 + .60625 \end{bmatrix} = \begin{bmatrix} .375 \\ .625 \end{bmatrix} = \mathbf{q}$$

∎

If the total population of the metropolitan region in Example 1 is 1 million, then the $\mathbf{q}$ from Example 4 would correspond to having 375,000 persons in the city and 625,000 in the suburbs. At the end of one year, the migration *out of* the city would be (.05)(375,000) = 18,750 persons, and the migration *into* the city from the suburbs would be (.03)(625,000) = 18,750 persons. As a result, the population in the city would remain the same. Similarly, the suburban population would be stable.

The next example shows how to *find* a steady-state vector.

EXAMPLE 5 Let $P = \begin{bmatrix} .6 & .3 \\ .4 & .7 \end{bmatrix}$. Find a steady-state vector for P.

Solution First, solve the equation $P\mathbf{x} = \mathbf{x}$.

$$P\mathbf{x} - \mathbf{x} = \mathbf{0}$$
$$P\mathbf{x} - I\mathbf{x} = \mathbf{0} \qquad \text{Recall from Section 1.4 that } I\mathbf{x} = \mathbf{x}.$$
$$(P - I)\mathbf{x} = \mathbf{0}$$

For P as above,

$$P - I = \begin{bmatrix} .6 & .3 \\ .4 & .7 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -.4 & .3 \\ .4 & -.3 \end{bmatrix}$$

To find all solutions of $(P - I)\mathbf{x} = \mathbf{0}$, row reduce the augmented matrix

$$\begin{bmatrix} -.4 & .3 & 0 \\ .4 & -.3 & 0 \end{bmatrix} \sim \begin{bmatrix} -.4 & .3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3/4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then $x_1 = \frac{3}{4}x_2$ and x_2 is free. The general solution is $x_2 \begin{bmatrix} 3/4 \\ 1 \end{bmatrix}$.

Next, choose a simple basis for the solution space. One obvious choice is $\begin{bmatrix} 3/4 \\ 1 \end{bmatrix}$ but a better choice with no fractions is $\mathbf{w} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ (corresponding to $x_2 = 4$).

Finally, find a probability vector in the set of all solutions of $P\mathbf{x} = \mathbf{x}$. This process is easy, since every solution is a multiple of the $\mathbf{w}$ above. Divide $\mathbf{w}$ by the sum of its entries and obtain

$$\mathbf{q} = \begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix}$$

As a check, compute

$$P\mathbf{q} = \begin{bmatrix} 6/10 & 3/10 \\ 4/10 & 7/10 \end{bmatrix}\begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix} = \begin{bmatrix} 18/70 + 12/70 \\ 12/70 + 28/70 \end{bmatrix} = \begin{bmatrix} 30/70 \\ 40/70 \end{bmatrix} = \mathbf{q}$$

The next theorem shows that what happened in Example 3 is typical of many stochastic matrices. We say that a stochastic matrix is **regular** if some matrix power P^k contains only strictly positive entries. For the P in Example 3, we have

$$P^2 = \begin{bmatrix} .37 & .26 & .33 \\ .45 & .70 & .45 \\ .18 & .04 & .22 \end{bmatrix}$$

Since every entry in P^2 is strictly positive, P is a regular stochastic matrix.

Also, we say that a sequence of vectors $\{\mathbf{x}_k : k = 1, 2, \ldots\}$ **converges** to a vector $\mathbf{q}$ as $k \to \infty$ if the entries in the $\mathbf{x}_k$ can be made as close as desired to the corresponding entries in $\mathbf{q}$ by taking k sufficiently large.

THEOREM 18

If P is an $n \times n$ regular stochastic matrix, then P has a unique steady-state vector $\mathbf{q}$. Further, if $\mathbf{x}_0$ is any initial state and $\mathbf{x}_{k+1} = P\mathbf{x}_k$ for $k = 0, 1, 2, \ldots$, then the Markov chain $\{\mathbf{x}_k\}$ converges to $\mathbf{q}$ as $k \to \infty$.

This theorem is proved in standard texts on Markov chains. The amazing part of the theorem is that the initial state has no effect on the long-term behavior of the Markov chain.

EXAMPLE 6 In Example 2, what percentage of the voters are likely to vote for the Republican candidate in some election many years from now, assuming that the election outcomes form a Markov chain?

Solution For computations by hand, the *wrong* approach is to pick some initial vector $\mathbf{x}_0$ and compute $\mathbf{x}_1, \ldots, \mathbf{x}_k$ for some large value of k. You have no way of knowing how many vectors to compute, and you cannot be sure of the limiting values of the entries in the $\mathbf{x}_k$.

The correct approach is to compute the steady-state vector and then appeal to Theorem 18. Given P as in Example 2, form $P - I$ by subtracting 1 from each diagonal entry in P. Then row reduce the augmented matrix:

$$[(P - I) \quad \mathbf{0}] = \begin{bmatrix} -.3 & .1 & .3 & 0 \\ .2 & -.2 & .3 & 0 \\ .1 & .1 & -.6 & 0 \end{bmatrix}$$

Recall from earlier work with decimals that the arithmetic is simplified by multiplying each row by 10.[1]

$$\begin{bmatrix} -3 & 1 & 3 & 0 \\ 2 & -2 & 3 & 0 \\ 1 & 1 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -9/4 & 0 \\ 0 & 1 & -15/4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The general solution of $(P - I)\mathbf{x} = \mathbf{0}$ is $x_1 = \frac{9}{4}x_3$, $x_2 = \frac{15}{4}x_3$, and x_3 is free. Choosing $x_3 = 4$, we obtain a basis for the solution space whose entries are integers, and from this we easily find the steady-state vector whose entries sum to 1:

$$\mathbf{w} = \begin{bmatrix} 9 \\ 15 \\ 4 \end{bmatrix}, \quad \text{and} \quad \mathbf{q} = \begin{bmatrix} 9/28 \\ 15/28 \\ 4/28 \end{bmatrix} \approx \begin{bmatrix} .32 \\ .54 \\ .14 \end{bmatrix}$$

The entries in $\mathbf{q}$ describe the distribution of votes at an election to be held many years from now (assuming the stochastic matrix continues to describe the changes from one election to the next). Thus, eventually, about 54% of the vote will be for the Republican candidate.

NUMERICAL NOTE

You may have noticed that if $\mathbf{x}_{k+1} = P\mathbf{x}_k$ for $k = 0, 1, \ldots$, then

$$\mathbf{x}_2 = P\mathbf{x}_1 = P(P\mathbf{x}_0) = P^2\mathbf{x}_0,$$

and, in general,

$$\mathbf{x}_k = P^k\mathbf{x}_0 \quad \text{for } k = 0, 1, \ldots$$

To compute a specific vector such as $\mathbf{x}_3$, fewer arithmetic operations are needed to compute $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ rather than P^3 and $P^3\mathbf{x}_0$. However, if P is small—say, 30×30—the machine computation time is insignificant for both methods, and a command to compute $P^3\mathbf{x}_0$ might be preferred because it requires fewer human keystrokes.

PRACTICE PROBLEMS

1. Suppose that the residents of a metropolitan region move according to the probabilities in the migration matrix of Example 1 and that a resident is chosen "at random." Then a state vector for a certain year may be interpreted as giving the probabilities that the person is a city resident or a suburban resident at that time.
   a. Suppose that the person chosen is a city resident now, so that $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. What is the likelihood that the person will live in the suburbs next year?
   b. What is the likelihood that the person will be living in the suburbs in two years?

[1] *Warning:* Don't multiply only P by 10. Instead, multiply the augmented matrix for equation $(P - I)\mathbf{x} = \mathbf{0}$ by 10.