MTH 420-520 Handout/worksheet on Euler-Lagrange equations

We will assume that all mathematical operations below are justified.

Some facts below are NOT complete!

Assume $f : \mathbb{R} \to \mathbb{R}$ is differentiable at 0. Recall

$$f'(0) = \lim_{t\to 0} \frac{f(t) - f(0)}{t}$$

Now let $L : \mathbb{R}^n$ be given, and $u, v \in \mathbb{R}^n$ be given, and define $f(t) = L(u + tv)$. Then, from chain rule,

$$f'(0) = \lim_{t\to 0} \frac{f(t) - f(0)}{t} = \lim_{t\to 0} \frac{L(u+tv) - L(u)}{t} = L'(u)v = \nabla L(u) \cdot v.$$

Next let $L : \mathbb{R}^n \times \mathbb{R}^n$ be given, and define $f(t) = L(u + tv, p + tq)$. Then, from chain rule,

$$f'(0) = \lim_{t\to 0} \frac{L(u+tv, p+tq) - L(u,p)}{t} =$$

Now assume $u, v, p, q : \mathbb{R} \to \mathbb{R}$ are functions, and define

$$f(t) = L(x, u(x) + tv(x), p(x) + tq(x)).$$

Compute

$$f'(0) =$$

Now define

$$\phi(u) = \int_0^L L(x, u(x), u'(x))dx \tag{1}$$

where $u, u' : (0, L) \to \mathbb{R}$ are *admissible* functions.

The first variation $\lim_{t\to 0} \frac{\phi(u+tv)-\phi(u)}{t}$ can be quickly seen as equal $f'(0)$. If we are seeking critical points of ϕ, we set it to zero

$$0 = \lim_{t\to 0} \frac{\phi(u+tv) - \phi(u)}{t} = f'(0) = \tag{2}$$

(Here we have assumed that the operation of moving the derivative from the outside to the inside of the integral is justified.) Further, a step of integration by parts applied to (2), under assumption that the boundary terms vanish, gives

$$=$$

Finally, applying the Fundamental Lemma of Variational Calculus we derive

$$= \tag{3}$$

Example: Consider $L(x, u, u') = \frac{1}{2}(u'(x))^2 + 5u(x)^2 - sin(x)u$. Let the set V of admissible functions be the set of smooth functions which satisfy u(0)=0,u'(L)=0.

The Euler-Lagrange equations (3) for the functional (1) are

$$-(u'(x))' + \qquad =$$