MTH 420-520 Handout for LAB 4:
Minimization of functions,
with and without (equality) constraints, and in subspaces

Some facts below need to be completed.

For a function $\phi(x)$, $x \in \mathbb{R}^n$, the first variation is

$$\lim_{t \to 0} \frac{\phi(x + ty) - \phi(x)}{t}.$$

Relate to the notion of directional derivative: ...

(I). Minimize $\phi(x) = \frac{1}{2}x^T Kx - f^T x$ over $V = \mathbb{R}^n$.
Assume $K \in \mathbb{R}^{n \times n}$ is spd.
Use the first variation to find critical point x for which

$$y^T(Kx - f) = 0, \quad \forall y \in V.$$

Confirm this point is indeed the minimizer of $\phi(x)$.
Solve $\boxed{}$.

(II). Application: minimize

$$\phi(x) = \frac{1}{2} \parallel Ax - b \parallel^2, \text{ for } x \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}.$$

Assume A is full rank and that $m \geq n$.
Find the solution by applying the method in (I).
Solve

(III). Minimize $\phi(x)$, $x \in \mathbb{R}^n$ subject to $g(x) = 0$.
Set up the Lagrangian

$$L(x, \lambda) = \phi(x) - \lambda g(x).$$

Find the stationary points of $L(\cdot, \cdot)$. Solve

(IV). Minimize $\phi(x) = \frac{1}{2}x^T Kx - f^T x$ over a subspace $W \subset V$.
Assume $K \in \mathbb{R}^{n \times n}$ is spd.
Use the first variation to find critical point $x \in W$ for which

$$y^T(Kx - f) = 0, \quad \forall y \in W.$$

Solve