LAB4: Minimization in Inner Product spaces and under constraints. Best approximations and projections

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All students do A, B, D or more if you wish. **Extra** *can be substituted for the "regular" parts of any problem.*

Recall that an orthogonal basis is one where $\langle w_i, w_j \rangle = 0$ for each pair of basis vectors/functions w_i, w_j. An orthonormal basis is an orthogonal one in which in addition the vectors/functions have unit length.

(A) Consider the function $\phi : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$\phi(x) := \frac{1}{2} x^T K x.$$

where K is the matrix from LAB1, problem (B), with spring constants $c_1 = 1, c_2 = 1, c_3 = 1$, and no gravity), subject to

$$g(x) = -x_1 + x_2 - 1 = 0.$$

The constraint can be interpreted as requiring a fixed distance between the masses.
(i) Find the minimum of $\phi(x)$ subject to $g(x) = 0$.
(ii) Draw the contour plots of ϕ, and plot the constraint.

You will likely use Lagrange multipliers. The mysterious λ should be interpreted as follows: it is a proportionality constant between the vectors $\nabla\phi(x)$ and $\nabla g(x)$ which must be parallel !

(iii) Plot these two gradient vectors on the same plot as (ii) to illustrate how the Lagrange multiplier works.

Extra: Find the minimum of $\phi(x)$ given above subject to being on a limacon (see LAB 2). Demonstrate graphically as above. Hmm. There may be a nonlinear problem to solve. Do your best to try and/or ask me how.

(B) Now consider a revised version of the example from class. Let $V = \mathbb{R}^2$, $v^* = [1, -2]^T$, and let $W = span\{[1, -1]^T\}$ be a subspace of V.

Find the best approximation w^* to v^* in the subspace W, i.e., the minimizer of the function

$$\phi(w) := \frac{1}{2} \parallel w - v^* \parallel^2 . \tag{1}$$

Solve it "by hand" following the example in class.

Now develop the general methodology, along the following steps.
(i) Denote by $A = [1, -1]^T$ the 2×1 matrix whose column space (the set of vectors spanned by its columns) is exactly W.
(ii) Now write $w^* = A\alpha$, i.e., w is parametrized and uniquely identified by the (scalar) α. Rewrite the definition of

$$\phi(w) = \phi(\alpha) = \frac{1}{2} \parallel A\alpha - v^* \parallel^2 .$$

(iii) Now solve the minimization problem as in II in handout and tell me which equation you must solve to find α.

(iv) Now develop a more general theory. Since we said that w^* is the projection of v^* onto W, we can write $w^* = P_W v^*$. Please write this projection matrix P_W in terms of matrix A. (Pay attention to the dimensions, singularity etc. of the matrices that form P_W).

$$P_W = \ldots$$

Extra: Now solve the same problem using IV from Handout. Recall you must have that all vectors in W must be orthogonal to the residual. Could it be that it is sufficient that the columns of A are orthogonal to the residual? You could also solve this problem using III from Handout.

(C) Extend the previous problem. Let $V = \mathbb{R}^3$, take $v^* = [1,1,1]^T$, and $W = span\{[1,0,0]^T, [0,1,1]^T\}$.
(i) Confirm that the two vectors form an orthogonal basis for W.
(ii) Find the projection w^* of v^* onto W, that is, its best approximation in the subspace W^*.
(iii) Find the projection matrix P_W.

(D) Consider the vector space $V = C[-1,1]$ and $W = P_2[-1,1]$, the subspace of polynomials of degree less than or equal 2 over the interval $[-1,1]$. Recall that V is an inner product space with the inner product $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx$.

A good basis for $P_2(-1,1)$ is $\{1, x, x^2\}$.
(i) Show that this basis is not orthogonal. Correct it by changing the last basis function to $1 - ax^2$ with a chosen so you get the orthogonality.
(ii) Find the coefficients of expansion for $f(x) = 1 - x + 6x^2$ in the orthogonal basis.
(iv) Now consider a function $f(x) = x^4$ which is in V but not in W. Find its projection f^* onto W (i.e., the best approximation to W). Plot $f(x)$ and the approximation $f^*(x)$.
Extra: can you identify the projection operator analogous to the projection matrix from the previous problems?

Integrals can be computed by hand or using MATLAB functions, for example quad. See the following example or use `doc quad`

```
%-- put this in an M-file
function y = myfun(x)
y = sin(pi*x).*sin(2*pi*x);
%--
%-- now compute the integral over (-1,1)
quad(@myfun,-1,1)
%
% or you can quickly code it as an inline function
myf = @(x)(sin(pi*x).*sin(2*pi*x));
quad(myf,-1,1)
```