MMAM (MTH 420/520) LAB5: Fourier series and analysis

All students in MTH 420 turn in A,B, and one of C or D (or both)

Fourier series on the interval $(-L, L)$ for a function $f(x)$ is the series

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi}{L}x) + \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi}{L}x). \tag{1}$$

Recall that the coefficients are given by

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{n\pi}{L}x)dx, n = 0, 1, \ldots \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{n\pi}{L}x)dx, n = 1, 2, \ldots.$$

Here are some clues on the behavior of Fourier series:

- A function that is smooth and has smooth derivatives has a Fourier series that converges uniformly. (*A Fourier series that has terms $O(\frac{1}{n^2})$ converges uniformly because the terms can be bounded by an absolutely convergent series of numbers).*
- A function that is only piecewise continuous converges at its points of discontinuity to the average value between the left and right limits. (*Such a function cannot have a uniformly convergent Fourier series. In particular, a series that has terms which are $O(\frac{1}{n})$ does not converge uniformly.*)
- The phenomenon when the Fourier series oscillates near the discontinuity of the original function is called *Gibbs phenomenon*. Identify it when it occurs.

`Watch the demonstration on how to construct Fourier series in MATLAB. We will use the Fourier series for the square wave function on` $(-\pi, \pi)$

$$f_0(x) = \begin{cases} 1, & 0 < x \\ -1, & x < 0 \end{cases} \approx F_0(x) = \sum_{n \text{ odd}}^{\infty} \frac{4}{n\pi}\sin(nx)$$

You can run `lab5fourier(3,pi)` to see $N = 3$ terms of the series $F_0(x)$ for $f_0(x)$ over $(0, pi)$. (Next try $N = 5, N = 15$ etc.)

(A) Match a given function $f_j(x)$ with one of $F_k(x)$, on $(-\pi, \pi)$, by experimenting. Provide evidence, discuss, but be concise (*No more than 2 pages and 6 graphs on this project*).

(f1)	$f(x) = x$
(f2)	$f(x) = e^{-\lvert x\rvert}$
(f3)	$f(x) = \pi^2 x - x^3$
(f4)	$f(x) = \sin^2(x)$
(f5)	$f(x) = \begin{cases} 1, & 0 < x < \pi/2 \\ -1, & -\pi/2 < x < 0 \\ 0, & \pi/2 < \lvert x\rvert < \pi \end{cases}$
(f6)	$f(x) = x^2$

(F1)	$F(x) = \frac{1}{2} - \frac{\cos(2x)}{2}$
(F2)	$F(x) = \frac{\pi^2}{3} + 4\sum_n \frac{(-1)^n}{n^2}\cos(nx)$
(F3)	$F(x) = \frac{e^\pi - 1}{\pi e^\pi} + \frac{2}{\pi e^\pi}\sum_n \frac{1}{n^2=1}(e^\pi - (-1)^n)\cos(nx)$
(F4)	$F(x) = 2\sum_n \frac{(-1)^{n+1}}{n}\sin(nx)$
(F5)	$F(x) = \frac{2}{\pi}\sum_n \frac{1}{n}(1 - \cos(\frac{n\pi}{2}))\sin(nx)$
(F6)	$F(x) = 12\sum_n \frac{(-1)^{n+1}}{n^3}\sin(nx)$

IDEAS FOR EXPERIMENTING: First, pick $j = 1, \ldots 6$ and plot the given function $f_j(x)$ on $[-\pi, \pi]$. What are its properties: (i) is it even, odd, or neither. (If even, use cosine series only. If odd, use sine series). (ii) Is the function continuous ? Is its derivative continuous ? (If smooth, use a series with high order of n in the coefficients. If not, use one with the smaller degree). (iii) Pick a k, and plot the first partial sums of Fourier series $F_k(x)$ to get an idea how the Fourier series behaves. Now you can identify f_j with one of F_k using the clues above (they are not rigorously written) and/or from the plots.

(**B**) For the function f_0 and its Fourier series F_0, give your opinion on how well the Fourier series F_0 with just a few N terms approximates f_0. In particular, compare their *values* of f_0 and F_0 at $x = 0, 0.1, 0.5, 1$ depending on how many terms N you use. *Now this may not be a fair comparison because from theory it is known that the "approximation" by Fourier series is optimal only in the Least Squares sense!* Collect the information about LSQ error depending on the number of terms N for $N = 2, 4, 6, 8, 10$. What do you think?

Extra: repeat for f_2 and its Fourier series that you found in (A).

(**C**) Now we follow the ideas developped in the handout and solve the heat/diffusion equation with the Fourier series.

(2)
$$u_t - u_{xx} = 0, \;\; x \in (0,1), \; t > 0$$

with the initial condition $u(x,0) = u_0(x) = 1$, and the boundary conditions $u(0,t) = 0 = u(1,t) = 0$ $(L = 1)$. Run `lab5fourier_heat(20,1,0.1)` to animate the solution until $t = 0.1$, using $N = 20$ terms, with $L = 1$.

- The operator $K = -\frac{d^2}{dx^2}$, with zero Dirichlet boundary conditions, has the eigenvalues $\lambda_n = (n\pi)^2$, and it has the eigenfunctions $sin(n\pi x)$.
- With the information about eigenvalues and eigenfunctions we can "diagonalize" K.
- We write u_0 in the Fourier basis, that is
$$u_0(x) = 1 = \sum_{n=1,odd}^{\infty} \frac{4}{n\pi} sin(n\pi x).$$
That is, the Fourier coefficients are $a_n = 0, \forall n$, and $b_n = \frac{4}{n\pi}$, for n odd, and $b_n = 0$, for n even. (To be precise, these are Fourier coefficients for the odd extension of $u_0(x)$ from $(0,1)$ to $(-1,1)$.)
- The solution to the heat equation (2) can be shown to be equal to

(3)
$$u(x,t) = \sum_{n\ odd}^{\infty} e^{-\lambda_n t} b_n \sin(n\pi x) = \sum_{n\ odd}^{\infty} e^{-(n\pi)^2 t} \frac{4}{n\pi} \sin(n\pi x)$$

Use the demo to animate your approximation to $u(x,t)$ at $t = 0$, $t = 0.01$, $t = 0.1$. (You can use as many terms of the Fourier series as you wish, but state how many you are using). (Do not turn in the plots, save the trees).

(i) Discuss the behavior. Find t_{STOP} for which the maximum $u(x, t_{STOP})$ on the interval $x \in (0,1)$ is above 0.7 and below 0.8. **Report the value t_{STOP} that you found and how you found it**.

(ii) Now write the solution when $u_0(x) = \sin(\pi x) + 0.5sin(3\pi x)$. (Hint: write what b_n and a_n are, then identify the corresponding λ_n. Next use (3) to write

$$u(x,t) =$$

Extra: implement these in the code, and plot the solution when t=0.1.

(**D**) The wave equation can also be solved with Fourier series. `lab5fourier_wave(20,1,5)` animates the vibrating spring until $t = 5$, using $N = 20$ terms, with $L = 1$.

As you can see, the code simply lets the function (vibrating string)

$$u(x,t) = \cos(\pi t)\sin(\pi x) + 0.5\cos(3\pi t)\sin(3\pi x)$$

come to life. **Show** that this function satisfies

(4)
$$u_{tt} = c^2 u_{xx},$$

for some c. What c? What are the two boundary conditions and the two initial conditions that $u(x,t)$ satisfies that you would need to solve the problem? For what t does the string have the same position as initially?