MTH 420/520/Peszynska. Midterm 2, Spring 2016

NAME:

Please show all relevant work to get full credit. Please show all steps of work by hand.

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(1) (5 points) Write the Euler-Lagrange equations which help to find the minimizer of $\phi(u) = \int_0^1 (\frac{1}{2}(u'(x))^2 - e^x u(x))dx$ satisfying $u(0) = 0 = u(1)$. Solve them.

(2) (5 points) Write the Euler-Lagrange equations which help to find the minimizer of $\phi(u) = \int_0^1 \frac{1}{2}\left((u'(x))^2 + 3(u(x))^2\right) dx$ satisfying $u(0) = 0 = u(1)$. Solve them.

(3) Find the function(s) $u(x)$ and the numbers λ satisfying

$$-u''(x) = \lambda u(x), x \in (0, \pi) \tag{1}$$

$$u'(0) = 0,\ u(\pi) = 0. \tag{2}$$

Note: there are many such numbers and many such functions. We are not interested here in the trivial solution $u(x) = 0$.

(4) (5 points) (A) Given $k > 0$, for what c does the function $u(x, t) = \exp(-ct)\sin(5\pi x)$ satisfy the heat equation $u_t - ku_{xx} = 0$ for $x \in (0, 1)$ and $t > 0$?
(B) What boundary conditions and initial condition does this solution satisfy? Describe the physical meaning of these initial and boundary conditions for the heat equation.
(C) Given (A-B), conjecture what would be the solution $v(x, t)$ if the initial condition was $v(x, 0) = \sin(10\pi x)$.
(D) Now that you have $u(x, t)$ and $v(x, t)$, verify that every linear combination $w(x, t) = au(x, t) + bv(x, t)$ (for any two constants $a, b \in \mathbb{R}$) satisfies the heat equation $w_t - kw_{xx} = 0$, with the same boundary conditions. (This property is called linearity).