MTH 451-551, Fall 2017, Assignment 6. Each problem is worth 5 points.

In this assignment you will work with iterative methods, estimates of eigenvalues and condition numbers. See class Handout on stationary iterative methods for solving $Ax = b$. You can also now practice your ability to calculate matrix norms and condition numbers.

1. (451 and 551) Solve 24.2 a.

2. (451 and 551) Consider the "discrete Laplacian" matrix $-\Delta_h = \frac{1}{h^2}A \in \mathbb{R}^{N\times N}$, with $h = \frac{1}{N+1}$ where A is given as

$$A = \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & \cdots & & \\ & & -1 & 2 \end{bmatrix} \tag{1}$$

that we discussed in class. (We also discussed its eigenvalues and eigenvectors).

(a) (451 and 551) Calculate the norms $\|\cdot\|_p$ for $p = 1, 2, \infty$ and the exact condition numbers κ_2 for the matrices A and $-\Delta_h$. How do they scale with h and N?

(b) (551) How does the smallest eigenvalue $\lambda_{min,h}$ of $-\Delta_h$ compare to π^2, the smallest eigenvalue of the continuous operator which $-\Delta_h$ approximates? (Hint: write the formula for $\lambda_{min,h}$ from class, and use calculus to show how $|\lambda_{min,h} - \pi^2|$ changes with h.).

3. (Theoretical) Consider $a \in \mathbb{R}$ and the matrix A

$$A = \begin{bmatrix} 2 & -1 & \\ -1 & a & -1 \\ & -1 & 2 \end{bmatrix}. \tag{2}$$

(a) Estimate the eigenvalues of A using the Gerschgorin theorem, and derive a sufficient condition on a that guarantees that A is positive definite. Compare to the condition on a that guarantees that A is (row) diagonally dominant (i.e., that in each row $A_{ii} \geq \sum_{i\neq j} |A_{ij}|$, and that at least one row has a strict inequality).

(b) Calculate G_{JAC} for the matrix A depending on a. Compare $\rho(G_{JAC})$ to $\| G_{JAC} \|_1$ and $\| G_{JAC} \|_\infty$, and find a sufficient condition for the convergence of Jacobi iteration.

(c) Predict how many iterations will be needed to decrease the error in the solution of $Ax = b$ using Jacobi method by the factor $tol = 10^{-5}$. (Does the answer depend on the choice of norm?).

4. (451-551, Computational) Confirm experimentally your findings in Problem 3b-c. If there is any discrepancy, discuss. In particular, can you weaken your sufficient condition? (That is, find an a for which the sufficient condition does not hold yet A is spd, and Jacobi method converges).
Hint on verifying Problem 3b: The code below can be used for a given matrix A to check if A is positive definite, and to calculate $\rho(G_{JAC})$.

```
min(eig(A))
D=triu(tril(A));M=D-A;GJAC=inv(D)*M;rho=max(eig(GJAC))
```

Hint on verifying Problem 3c) Consider $x = [1, 1, 1]^T$ as the true solution to $Ax = b$, and iterate from initial guess $x^{(0)} = [0, 0, 0]^T$ to approximate the solution by Jacobi iteration $x^{(k+1)} = G_{JAC}x^{(k)} + c$.

```
x=[1,1,1]'; b=A*x;xg0=0*b;c=D\b;iter=0;xg=xg0;
iter=iter+1,xg=GJAC*xg+c,norm(xg-x,inf)/norm(xg0-x,inf),
iter=iter+1,xg=GJAC*xg+c,norm(xg-x,inf)norm(xg0-x,inf),
....
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(551 students should also consider norms other than ∞ when checking 3c).

5. (Extra: computational). Repeat 3-4 for Gauss-Seidel method, and compare to Jacobi method.
Find (experimentally) the optimal ω depending on a and matrix A from (2) for the SOR method which guarantees that the method converges. Experiment also with matrix $-\Delta_h$ from (1), depending on h or N. You can also experiment with the Richardson's method to find the optimal α.