MTH 452-552 Assignment 8. **FirstName LastName**
Prof. M. Peszynska Due: 03/20/2018
Or earlier!!!

Problem 0, warm-up (do not turn in). Reproduce plots and convergence plots from class notes. Use $\|\cdot\|_\infty$ in your convergence studies.
`Code templates are available in class notes.`

Problem 1, theoretical. [552 do both, 452 do at least one of (a,b).]
(a) Consider a Dirichlet BVP for $-u'' + u' = f$. Propose two variants of a discrete scheme so that the truncation error is either $O(h^2)$ or $O(h)$.
(b) Consider approximating $u''(t)$ on a nonuniform grid using values

$$u(t), u(t-a), u(t+b), \quad a, b > 0,\ a \neq b.$$

Come up with $D^{2,mine}_{a,b} u$ which is as accurate as possible. (Show the analysis of course).

Problem 2, computational. [452 must do at least (a), 552 do both]
(a) Implement and test convergence of your code for BVP

$$-u'' = f,\ t \in (0,1),\ \ u(0) = 0, u(1) = 0. \tag{1}$$

Test when (i) $f = 1$, and (ii) $f = H(t-0.5) = \begin{cases} 0, & t < 0.5 \\ 1, & t \geq 0.5 \end{cases}$.
Did you get what you expected? Explain the behavior of the error.
Hint: to find the exact solution for (ii), solve the equation piecewise. Glue together at $t = 1/2$ by requiring continuity of u and of u' at that point. (Clearly this means we do not expect the ODE to hold at $t = 1/2$, so we mean here a relaxed notion of solutions). When testing convergence, consider the family of grids starting with $h = 1/10$, and another with $h = 1/11$.
(b) Now extend the code to work for

$$-\varepsilon^2 u'' + u = 1, t \in (0,1),\ \ u(0) = 0, u(1) = 0. \tag{2}$$

Set $\varepsilon = 1$, and find h, if possible. so that the error is less than 10^{-3}. Repeat for $\varepsilon = 10^{-1}$. (Extra: consider an even smaller ε.) Comment on your findings.
Hint: You can easily find the exact solution. If you prefer, you can also use a fine grid solution as a proxy.
This problem is known as a singularly perturbed problem, and the solution for small ϵ exhibits a boundary layer, typically for fluid flow simulations.

Extra project 8: Propose and implement variable grid strategy for (2). Use intuition to come up with the best variable grid possible when $\varepsilon = 10^{-1}$.
("Best grid" corresponds to "the smallest number of grid points with the smallest error").