MTH 453-553 Assignment 1. **FirstName LastName**
Prof. M. Peszynska Due: 04/13/2018

`Use the code templates for Dirichlet BVP from class notes (modify as needed).`

Problem 1. Confirm $O(h^2)$ convergence in $p = 2, \infty$ grid norms for

$$-u'' = f,\ x \in (0,1), u(0) = u_a, u(1) = u_b, \tag{1}$$

with Dirichlet BC.
Compare to the case when at $x = 1$ we impose Neumann condition $u'(1) = g_b$

$$-u'' = f,\ x \in (0,1), u(0) = u_a, u'(1) = g_b. \tag{2}$$

Study and compare the behavior of the error. Consider true solutions (i) $u(x) = e^x$ and when (ii) $u(x) = \sin(\pi x)$, and for each, manufacture appropriate data for the Dirichlet case (1) and for the mixed case (2). You should use the one-sided approximation to the derivative in the Neumann condition in (2). (Higher order approximations can be also considered for extra credit). Compare to the textbook Sec. 2.12.

Problem 2. Consider the problem $-\varepsilon u'' + u' = 1$, with homogeneous Dirichlet BC. (See textbook Eq. (2.90)).
Implement a finite difference approximation to this problem. In particular, consider the one-sided upwind and one-sided downwind approximation to the u' term. (Students in 553 also implement the two-sided approximation).
Report and discuss the convergence of the method in $p = 2, \infty$ grid norms for $\varepsilon = 10^{-1}, 10^{-2}$ when varying h. You should make sure to include the grid for which the $p = \infty$ error is less than 10%.

Problem 3, theoretical, do not turn in. Consider the problem $-(ku')' = 0$ on $(0, 1)$, with $u(0) = 0, u(1) = 1$. Assume $k(x)$ is given.
(i) Find the solution to this problem. (Hint: integrate the equation twice).
(ii) The smoothness of the solution depends on the smoothness of k. Discuss the details.
(iii) Now consider the case when $k(x)$ is piecewise constant, with a jump from value 1 to 10 at x=0.5. (This case arises frequently in applications with heterogeneous materials.) Solve (i) and revisit (ii).
(iv) What grid and approach would you recommend if someone wanted to implement a FD method for the problem in (iii)?

Problem 4, theoretical, do not turn in. (i) Find the Green's function for the mixed Dirichlet-Neumann problem $-u'' = f$ on $(0,1)$, with $u(0) = 0, u'(1) = 0$.
In other words, consider a given point $x_0 \in (0,1)$, and solve the problem $-G'' = 0$ on $(0, x_0)$ and in $(x_0, 1)$, with the boundary conditions $G(0) = 0, G'(1) = 0$. (Since the solution depends on x_0, we can denote it by $G_{x_0}(x)$ or $G(x_0; x)$.) Assume that G is continuous at x_0 and that $G'|_{x_0}$ takes a jump of -1, so that we can write, informally, $-G'' = \delta_{x_0}$.
(ii) Now estimate $\max_{x_0} \int_0^1 G(x_0; x)dx$; this can be used to establish the stability in ∞ norm of the FD solution to (2).
(iii) What do you recommend if someone were to implement FD method for this problem?