**MTH 453-553 Assignment 1.** **FirstName LastName**
Prof. M. Peszynska Due: 04/14/2018

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`Use the code templates for Dirichlet BVP from class notes (modify as needed).`

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**Problem 1.** Confirm $O(h^2)$ convergence in $p = 2, \infty$ grid norms for

$$-u'' = f,\ x \in (0,1), u(0) = u_a, u(1) = u_b, \tag{1}$$

with Dirichlet BC.
Compare to the case when at $x = 1$ we impose Neumann condition $u'(1) = g_b$

$$-u'' = f,\ x \in (0,1), u(0) = u_a, u'(1) = g_b. \tag{2}$$

Study and compare the behavior of the error. Consider true solutions (i) $u(x) = e^x$ and when (ii) $u(x) = \sin(\pi x)$, and for each, manufacture appropriate data for the Dirichlet case (1) and for the mixed case (2). You should use the one-sided approximation to the derivative in the Neumann condition in (2). (Higher order approximations can be also considered for extra credit). Compare to the textbook Sec. 2.12.

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**Problem 2.** Consider the problem $-\varepsilon u'' + u' = 1$, with homogeneous Dirichlet BC. (See textbook Eq. (2.90)).
Implement a finite difference approximation to this problem. In particular, consider the one-sided upwind and one-sided downwind approximation to the $u'$ term. (Students in 553 also implement the two-sided approximation).
Report and discuss the convergence of the method in $p = 2, \infty$ grid norms for $\varepsilon = 10^{-1}, 10^{-2}$ when varying $h$. You should make sure to include the grid for which the $p = \infty$ error is less than 10%.

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**Problem 3, theoretical, do not turn in.** Consider the problem $-(ku')' = 0$ on $(0, 1)$, with $u(0) = 0, u(1) = 1$. Assume $k(x)$ is given.
(i) Find the solution to this problem. (Hint: integrate the equation twice).
(ii) The smoothness of the solution depends on the smoothness of $k$. Discuss the details.
(iii) Now consider the case when $k(x)$ is piecewise constant, with a jump from value 1 to 10 at x=0.5. (This case arises frequently in applications with heterogeneous materials. ) Solve (i) and revisit (ii).
(iv) What grid and approach would you recommend if someone wanted to implement a FD method for the problem in (iii)?

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**Problem 4, theoretical, do not turn in.** (i) Find the Green's function for the mixed Dirichlet-Neumann problem $-u'' = f$ on $(0, 1)$, with $u(0) = 0, u'(1) = 0$.
In other words, consider a given point $x_0 \in (0, 1)$, and solve the problem $-G'' = 0$ on $(0, x_0)$ and in $(x_0, 1)$, with the boundary conditions $G(0) = 0, G'(1) = 0$. (Since the solution depends on $x_0$, we can denote it by $G_{x_0}(x)$ or $G(x_0; x)$.) Assume that $G$ is continuous at $x_0$ and that $G'|_{x_0}$ takes a jump of $-1$, so that we can write, informally, $-G'' = \delta_{x_0}$.
(ii) Now estimate $\max_{x_0} \int_0^1 G(x_0; x)dx$; this can be used to establish the stability in $\infty$ norm of the FD solution to (2).
(iii) What do you recommend if someone were to implement FD method for this problem?