MTH 453-553 Assignment 2. **FirstName LastName**
Prof. M. Peszynska Due: 04/20/2018

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You can use the code templates for Poisson's problem from the class notes (amend
them and modify as needed, and show what you did)
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Problem 0: warm-up: do not turn in. Confirm $O(h^2)$ convergence in $p = 2, \infty$ grid norms for the FD solution to

$$-\Delta u'' = f,\ (x, y) \in (0, 1)^2,\ u|_{\partial\Omega} = 0. \tag{1}$$

Assume true $u(x, y) = \sin(\pi x)\sin(\pi y)$ is known. (Do not turn in).

Problem 1. A student is attempting to verify $O(h^2)$ convergence in $p = 2, \infty$ grid norms for the problem (1). Motivated by the 1d examples, they use the following code

```
function v=rhsfun(x,y)
    v = 2;
end
function v=exactfun(x,y);
    v = x*(1-x)/2 + y*(1-y)/2;
end
```

The solution at a first glance "looks" (almost) all-right. Yet, the error is always about 0.1, and does not change when they refine the grid. What went wrong? Suggest how to correct and check convergence.

Problem 2. Consider the unit square $\Omega_0 = (0, 1) \times (0, 1)$, and a lattice of $\mathcal{L} := N \times N$ uniformly spaced points which covers its closure. (553 use $N = 11$ or larger, 453 can use $N = 5$).

On this grid draw a non-convex connected domain $\Omega \subset \Omega_0$ with a boundary $\partial\Omega$ made of segments parallel to the x or y axis and drawn by connecting the lattice points; denote $\mathcal{O} := \Omega \cap \mathcal{L}$. The domain should depict a letter, or an "image" of something recognizable to your grandmother (You can consult Her). Make sure that $\mathcal{O} \neq \emptyset$.

Solve the homogeneous Dirichlet Poisson's problem (1) with $f \equiv 1$ on this domain.

Your solution is the surface or contour plot; you can interpret it as steady-state temperature of a region uniformly heated from below, immersed in a bath of constant temperature.

Hint: the simplest way to organize the grid by hand is to patiently number the boundary nodes ($\mathcal{B} = \partial\Omega \cap \mathcal{L}$) and those outside Ω (collect them in the set $\mathcal{O}^C$). In the code, apply "brute force": treat the nodes in $\mathcal{O}^C$ the same way as those in $\mathcal{B}$. (While this is not efficient, it will do the job). More elegant solutions involve rewriting the code.

Extra credit: We do not know the true solution. To check convergence and error $u - u_h$, you can resort to a fine grid solution used as a proxy for the true solution, $u_{fine} \approx u$. Consider a sequence of lattices $\mathcal{L}_N$ with $N = 11, 101, \ldots N_{fine}$ so that the shape of Ω is unchanged when represented on all the lattices $\mathcal{L}_N$.

Problem 3, theoretical, do not turn in. Discover the tyranny of scales by doing some independent reading, e.g., of Chapter 4 of the textbook.
(a) Assume your linear solver for matrix $A \in \mathbb{R}^{N \times N}$ corresponding to the discrete form of (1) scales like $O(N^{1+\alpha})g(N)$, where $g(\cdot)$ is some increasing function of N. Estimate the number of flops needed to solve the Poisson's problem in 3d with a given $M_x \times M_y \times M_z$ grid. Then refine the grid by a factor of 2 in each direction, and compare the complexity.
(b) Now get α and $g(\cdot)$ for the linear solver from the families as below. Include the use of a direct (banded or dense) linear solver, and of an iterative linear solver such as CG or Jacobi. (You can refer to Chapter 4 of textbook for information). Prepare the estimate of complexity for these known choices of $\alpha, g(\cdot)$.
Extra credit: synthesize your knowledge and prepare a table for (b) with a concise summary of your findings.

Problem 4, theoretical, do not turn in. Consider given constants a, b, c and a PDE

$$au_{xx} + 2bu_{xy} + cu_{yy} + du_x = f. \tag{2}$$

Propose a FD discretization, and check its LTE.
Check the type of the PDE (2). (textbook E.1.1). What conditions (initial, boundary) are needed to solve the problem if
(i) $a = 10, b = 1, c = 5, d = 1$,
(ii) $a = 0, b = 100, c = 0, d = 0$,
(iii) $a = 1, b = 0, c = -100, d = 0$,
(iv) $a = 1, b = 0, c = -100, d = 1$.
Propose examples of such conditions and how to implement them in the numerical algorithm.

Problem 5, theoretical, do not turn in. Consider the discretization of $-u'' = f, x \in (a, b)$ with a Dirichlet condition at $x = a$, and Neumann or Robin condition at $x = b$. Propose a FD method to discretize the problem so that the system to be solved has the form as in (C.24) in textbook. Now write out the eigenvalues of the matrix A following (C.25). Is this enough to prove the stability of the method in 2-norm? (If yes, get the estimates for $\| A^{-1} \|_2$) Compare to the Greens' function approach from HW1.