MTH 453-553 Assignment 4. **FirstName LastName**
Prof. M. Peszynska Due: 05/TBA/2018

`Show the code you develop for schemes other than upwind.`

Consider the IVP for advection equation, with $a \in \mathbb{R}$

$$u_t + a u_x = 0, \;\; u(x,0) = u_0(x). \tag{1}$$

Consider the initial data: (a) the smooth initial data $u_0(x) = \exp(-100(x-0.3)^2)$ and (b) the Riemann data $u_0(x) = 1 - H(x)$.) (In experiments use $a = 1$, $0 \le t \le 0.5$. In (a) $x \in (0,1)$, periodic boundary conditions. In (b) use $-0.5 \le x \le 1$ and the left boundary condition $u(-0.5,t) = 1$).

Problem 1: convergence of smooth and non-smooth solutions (implementation). Determine the accuracy experimentally in p-grid norm at $t = 0.5$ depending on ν of the following methods: upwind, Lax-Friedrichs, and Lax-Wendroff (Beam-Warming for extra credit). Use p-grid norm: (a) compare $p = \infty$ and $p = 2$ (the second predicted by theory), for (b) compare $p = 2$ and $p = 1$ (the first not covered by theory in this class, the second covered by theory not discussed in this class). Choose "perfect" ν^{perfect} to minimize numerical diffusion or dispersion, and at least one non-perfect $\nu = 0.5\nu^{\text{perfect}}$.
Extra: try downwind just to see how bad things go.

Problem 2: analyze accuracy and stability (theory only). Derive the LTE, the modified equation, and check stability of the scheme (at least first two.)

(1) a scheme combining FE in time with central difference in space (unstable, explain why via modified equation and via von-Neumann)
(2) a scheme combing FE in time with downwind scheme (unstable, big time)
(3) a scheme combing BE in time with upwind scheme (very diffusive, explain why)

Be concise. If you are so lucky to find a reference, state the result and cite the reference, but do not copy the calculations.

Problem 3: Bonus: mixture of experimental and theoretical problem solving. It is known that the exact solution to $u^D_t + a u^D_x = D u^D_{xx}, x \in \mathbb{R}$ with the Riemann initial data as in (b) is given by

$$u^D(x,t) = \frac{1}{2}\left(1 - \text{erf}\left(\frac{x - at}{\sqrt{4Dt}}\right)\right), \tag{2}$$

while the solution to $u_t + a u_x = 0$ is the travelling wave $u(x,t) = u_0(x - at)$.
(One can prove that $\| u(\cdot,t) - u^D(\cdot,t) \|_{L^1} = O(\sqrt{Dt})$*).*
Identify D for each of the upwind and Lax-Friedrichs schemes depending on $\Delta t, h$. Compare your numerical solution $(U^n_j)_{j,n}$ to (1) to its true solution $u(x,t)$, and to $u^D(x,t)$.
Is it indeed true that the numerical solution is closer to $u^D(x,t)$ than to $u(x,t)$? Quantify.
Which of the schemes: upwind or Lax-Friedrichs, is more diffusive?
Revisit Pbm 1b with the new interpretation, and provide a more accurate solution.