MTH 4/553 Midterm; May 4/2018. **Name:**

Books and notes are allowed. No electronics!

Pbm 1, 5 pts. (LTE) [453 choose at least one, 553 do at least two. Extra pts:1-2

(A) Propose a scheme to discretize $u_{xy} = f$. Calculate the LTE.

(B) Consider the heat equation $u_t - u_{xx} = f$; approximate it via MOL. Then discretize in time using the mid-point scheme. What is the LTE?

(C) Consider the equation $-(k(x)u_x)_x = f$; approximate it with a FD scheme. Calculate the LTE (you will need assumptions on the smoothness of $k(x)$). What happens if these do not hold?

Pbm 2, 5pts (Linear solver issues). [Solve the case for $k = 2$. Extra (1pt): do also $k = 3$.]

Consider $-\Delta u = f$, in $\Omega = (0, 1)^k \subset \mathbb{R}^k$. After discretization with a finite difference scheme on a $Mx \times My \times Mz$ ($Mz = 1$ for $k = 2$), we have the linear system $AU = F$. Assume it is solved with (A) a direct solver such as LU factorization (Gauss elimination), or (B) a banded direct solver, or (C) Conjugate Gradient, or (D) Multigrid. Provide an estimate of the increase in the computational cost when the grid is refined from $Mx \times My$ to $2Mx \times 5My$, while keeping Mz fixed.

You can assume the complexity is (A) $O(N^3)$, (B) $O(Np)$, (C) $O(N\#iters)$, or (D) $O(N)$. You have to come up with an estimate of (B) the band size p; and (C) $\#iters$, the number of iterations needed for CG to converge. (Refer to Chapter 4 of text).

Pbm 3, 5pts. (Well-posedness, also of discrete system). [Do either (A) or (B). Both (A-B) for extra credit (2pts)]

Consider $u'' = 1$, on $(0, 3)$, with (A) periodic, and (B) homogeneous Neumann bc.

(i) Is there a unique solution?

(ii) Now discretize with $h = 1$ (two interior points). Write the discrete system (use first order one-sided approximation for the derivatives in the boundary conditions). Discuss whether the discrete system has a solution; compare with your answer in (i).

Pbm 4, 5pts. (Stability via method of lines). [453 do A-B, 553 do A-C]

Consider the reaction-diffusion equation $u_t - ku_{xx} + ru = 0$, on $(0, 1)$, with $k = const > 0$, homogeneous Dirichlet bc, and some initial data.

(A) Write a semi-discrete system that needs to be solved for every t.

(B) Apply a single-step explicit time discretization method, and write a fully discrete equation in the matrix-vector form. Derive the condition on Δt that will make the scheme Lax-Richtmyer stable. You can assume $r > 0$, or work without this assumption for extra credit (1pt).

(C) Instead of B, define a fully discrete scheme with which the treatment of the reaction term ru is explicit in time. The scheme should be chosen so that when $r = 0$ the scheme should be unconditionally stable. Analyze the stability for general $r > 0$. (The case of $r \in \mathbb{R}$ is for extra credit, 2pts).