MTH 621/Peszynska/Fall 2015, Assignment 3

Please show enough of your work to justify the answer but be concise.
Use proper mathematical notation.

Solve 1, 6, 7, 8 or more for extra credit. In particular, solving 2-3-4-5 will give you excellent practice on Fourier series.

1. [**pts**] Consider the Hilbert space V and its subspace K as given. Find the best approximation in K of a given v.
   i) Let $K = \{v(x) = \alpha x^2; \alpha \in \mathbb{R}\}$ for $v(x) = x$ and $V = L^2(-1,1)$.
   ii) Let $K = \{v(x) = \alpha sin(x) + \beta cos(2x); \alpha, \beta \in \mathbb{R}\}$. $v(x) = sin(2x) - 3cos(x)$ and $V = L^2(-\pi,\pi)$.
   The solutions are actually quite simple so spend some time on establishing the proper background for this problem. For example, do we know that K is indeed a subspace of V; what is the orthonormal basis for K and V. Is $v \in V$ etc. ?

2. [**pts**] Consider $f(x) = x$, $g(x) = |x|$, and $h(x) = \begin{cases} 1, & x > 0 \\ 0, & x \leq 0 \end{cases}$, each on $(-\pi,\pi)$.
   For each of these functions, consider their periodic extensions, and determine what you would expect from the convergence of its Fourier series (without calculating it). Include the considerations of mean-square convergence, and pointwise and uniform convergence of the series combined with the discussion of the (lack of) continuity of the function and its derivatives. Refer to the theory from [GLee, Chapter 3] or from the class handout.

3. [**pts**] For functions in the previous problem, find their Fourier series and revisit the convergence issues. BE CONCISE. (I do not need to see the details of all the calculations).
   Extra: plot the first few partial sums).

4. [**pts**] Solve 3-2.2b. Use the series that you found to show
$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

5. [**pts**] Solve 3-3.1

6. [**pts**] Solve 3-2.3
   Important: this function is the so-called Green's function.

7. [**pts**] Consider the formal travelling wave solution to the homogeneous wave equation with "sine" initial displacement $u(x,o) = sin(x)$ and zero initial velocity, with $c = 1$, and show that the total energy
$$E(t) = \frac{1}{2} \int_{\mathbb{R}} \left((u_t)^2 + (u_x)^2\right) dx \qquad (0.1)$$

does not change. (For the energy to be finite, choose the support of the "sine" initial data to be compact, e.g., in $(0, \pi)$).
Note: the first part of the integral gives the kinetic energy, and the second the potential energy.
Challenge/extra: show that the above is true for any compactly supported initial data. (Hint: consider $\frac{dE}{dt}$ and integrate by parts several times). If you solved this part, you do not have to solve the first part again.

8. [**pts**] Solve 4-2.11 (a-d).
Extra: (e-f). You can use any analytical or computational method of your choice; include the references if any.