MTH 622/Peszynska/Winter 2009
Assignment 3
Please show all your work. Use proper mathematical notation.

1. The need to solve problems in variational sense rather than in classical sense arises when f or the coefficients are not smooth.
   Compute the weak solution to $-(k(x)u')' = 0$ on $(0, 1)$ with boundary conditions $u(0) = 0, u(1) = 1$ for $k(x) = \begin{cases} k_1, & x < 1/2 \\ k_2, & x \geq 1/2 \end{cases}$ where k_1, k_2 are two positive possibly distinct constants
   Determine the smoothness of this solution. (for what α, m, can we say that $u \in C^\alpha(0, 1), u \in H^m(0, 1)$?)

2. Consider the problem $-u" = f,\ x \in (0, 1)$ with mixed boundary conditions $u(0) = 0, u'(1) = 0$. Show that Poincaré-Friedrichs inequality is valid for any function in the space $V = \{v \in H^1(0, 1) : v(0) = 0\}$. Formulate an appropriate variational form for this problem and discuss its equivalence (if appropriate) with the classical problem.

3. (Follow up on Problem 1 from HW2: no credit): verify to which of the H^m spaces does the function $u(x) = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$ belong.

4. (Practice problem: no credit) For each of the following forms, determine if they are symmetric, continuous, and elliptic, coercive, on a given space (find the ellipticity/coercivity and continuity constants)

$$a(u, v) = \int_0^1 ku'v' + cuv, \quad \text{on } H^1(0, 1),$$

   with k, c piecewise constant as in Problem 1.

$$a(u, v) = \int_a^b (x^2 + 1)u'v', \quad \text{on } H_0^1(a, b)$$

$$a(u, v) = \int_0^1 (x^2)u'v' + uv, \quad \text{on } H_0^1(0, 1)$$

$$a(u, v) = \int_{-1}^1 (x + 2)u'v' + xuv, \quad \text{on } H^1(-1, 1)$$

$$a(u, v) = \int_a^b u'v' + Pu'v + cuv, \quad \text{on } H^1(0, 1)$$

   where P and c are nonnegative constants,

$$a(u, v) = \int_0^1 uv, \quad \text{on } H_0^1(0, 1)$$

   (Your answer will depend on the interval, on the coefficients, and on the space).

5. (Follow up on problem 4 from HW 2: no credit). Consider the form $a(u,v) = v^T A u$ with matrix A as in HW2, #4. What are the properties of this form ? Is there a unique solution to $Au = f$? How about to $a(u,v) =< f, v >$, $\forall v V = \mathbb{R}^2$? Are we guaranteed that the stationary point of the functional J_A you obtained for this matrix in HW2 is a minimizer ?

   Consider now $B = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$. Discuss the same issues as above but now associate the forms and the functional with matrices B, C, D instead of with A.

6. (Extra) Find the Riesz representer for δ in $H_0^1(-1,1)$ using the inner product $\int_{-1}^{1} u'v' + uv$.