MTH 622/Peszynska/Winter 2009
Assignment 4
Please show all your work. Use proper mathematical notation.

1. Consider an Euler differential equation

$$x^2y'' + mxy' + y = 0$$

where m is a real constant.
Find the general solution $y = y(x)$ using the substitution $z = \ln x$ to transform it to an ODE with constant coefficients, which you can solve with the usual Ansatz. Your answer will depend on m.

2. Find the function harmonic on the unit quarter disk (interesection of the disk with the first quadrant) and satisfying the homogeneous Dirichlet condition on $x = 0, y = 0$, and the Neumann condition $\frac{\partial u}{\partial r} = 1$ on the curvilinear part of the boundary.
Extra: plot the solution.

3. Find the function harmonic in the square $(0, \pi) \times (0, \pi)$ which satisfies homogeneous Neumann boundary conditions on the South and North boundaries, homogeneous Dirichlet condition on West side, and the condition $u(\pi, y) = \cos^2(y) = \frac{1}{2}(1 + \cos(2y))$ on the East side.
Extra: plot the solution.

4. Find the function harmonic on the unit cube which satisfies the conditions $u(x, y, 0) = \sin(\pi x)\sin(2\pi y), 0 \leq y \leq 1$ and $u(x, y, 1) = \sin(4\pi x)\sin(\pi y), 0 \leq y \leq 1$ and the homogeneous Dirichlet conditions on all other sides. **Hint:** you can follow example [S 6.2.2] to derive a general solution on a cube. In order to deal with non-homogeneous conditions on two sides, use linearity. What is the maximum value of that function on the cube ?
Extra: plot the solution.

5. (**Extra:**) This problem provides examples of functions in $H^1(\Omega)$ which has a singularity at the origin. Verify $u \in H^1$ and determine the kind of singularity (lack of continuity of the function or of its derivatives).
i) Ω to be unit disk, $u = \ln(\ln(\frac{2}{r}))$
ii) Ω is a disk of a small enough radius (how small ?), $u = \ln(|\ln(r)|)$
iii) Ω is the unit sphere, $u = \ln(r)$.
iv) Ω is the unit ball in $\mathbb{R}^d, d = 1, 2, 3$, $u = r^{\beta_d}$, with the constants such as $\beta_1 \geq 3/4$, $\beta_2 \geq 0$, $\beta_3 \geq -1/2$.