MTH 622/Peszynska/Winter 2016, Assignment 4. Part B

Please show enough work to justify your answer.
Use proper mathematical notation.

(1) Find the Euler-Lagrange equations for the functional

$$J(x,u,u') = \int_0^1 ((u')^2 sin(u) - x^2 u)dx$$

on some $V \subset C^1(0,1)$. (Do not attempt to solve them).

(2) Find the best Poincaré-Friedrichs constant for $\Omega = (0,1)$ as outlined in class, by minimizing the functional $J(u) = \int_\Omega (u')^2 dx$ subject to the constraint $\int u^2 dx = 1$.

(3) Find the distributional derivatives $\partial^k f$ for the regular distribution on $\Omega = (-1,1)$ identified with the function $f(x) = x^2,\ x \geq 0$, and $f(x) = x^3,\ x < 0$. Try $k = 1, 2, \ldots$ until you get the first singular distribution.

(4) Let $a \in R$.

(i) For what a can you show that $(u,v) = \int_0^1 au'v' + \int_{-1}^0 u'v'$ is an inner product on $H_0^1(-1,1)$? (Show it).

(ii) Find the Riesz representer $w_a(x) \in H_0^1(-1,1)$ with respect to this inner product for $J(v) = \int_{-1}^1 v(x)dx$.

Extra: (iii) For what a is $w_a \in H_0^1(-1,1) \cap H^2(-1,1)$?

(5) Consider the functional $J(u) = \int_0^1 [(u')^2 - u]dx$ on $H_*^1(0,1) = \{v \in H^1(0,1) : v(0) = 0\}$. (i) Find the variational formulation for its minimizer, from the definition of the first variation (consider $J(u + tv)$, and so on). (ii) Find this minimizer.

(6) **Extra:** For what α is $u(r,\theta) = r^\alpha$ in $H^1(B_0(1))$ in $d = 2$?

(7) For what a, with $\Omega = B_0(a)$, is $u(r,\theta) = log(r) \in W^{1,1}(\Omega) \in \mathbb{R}^d$ when $d = 2$?

Extra: What about when $d = 3$? Is $u \in W^{1,2}(\Omega)$?

(8) Show that $u(r,\theta) = loglog\frac{2}{r} \in H^1(B_0(1))$.