MTH 655, Winter 2017, Assignment 1

In this HW we revise fundamental concepts from differential equations and finite difference methods. We use a simple BVP

$$-\varepsilon u'' + u = 1, \quad x \in (0,1) \tag{1a}$$
$$u(0) = 0, \quad u(1) = 0 \tag{1b}$$

as our major example, with $\varepsilon > 0$. This problem is called "singularly perturbed" because, as $\varepsilon \to 0$, the solution becomes closer to $u \equiv 1$ except close to the boundary where it is forced to obey the boundary conditions.

For experts, and/or for extra credit, consider dealing instead with

$$-\varepsilon u'' + u = 1, \quad x \in (0,1) \tag{2a}$$
$$u(0) = 0, \quad u'(1) = 0 \tag{2b}$$

(Of course, you have to change the code and everything else).

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1. Analytical solution

Find the analytical solution to (1), and plot it (MATLAB?) for several different values of ε; comment on the behavior. (At most one page on this problem).

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2. Finite difference solution

Experiment with the finite difference solution for this problem. You can use the template `fd1d_singular.m` provided. It does not have the true solution coded in it, so you will have to do this based on Pbm 1. Once you have the true solution, implement finding an error. (See how this is done in `fd1d.m` which solves a different problem). The error for a finite difference solution is, for a given h,

$$e(h) = \max_j |u(x_j) - u_j| \tag{3}$$

For a nice and smooth problem, the error should be of second order, that is of $O(h^2)$.

- 2a. Plot the solution and numerical solutions for the case, as shown in the hand-out.
- 2b. Prepare a table of errors as follows.
  For $\varepsilon = 1$, report the error $e(h)$ for $h = 1/5, 1/10, 1/20, 1/50, 1/100$. What is the order of convergence?
  Same for $\varepsilon = 10^{-1}, 10^{-2}, \ldots$. At which point do you see that the order of convergence is no more *two* ?

You can also plot the errors on the loglog scale to see the slope which should be the order α if you expect $e(h) = O(h^\alpha)$

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3. Variational formulation

Set-up variational formulation of the problem (1) and (2).

4. Norms in functional spaces

Calculate $\| u \|_{L^2(0,1)}$, if possible, for

$$u(x) = 1, \tag{4}$$

$$u(x) = x, \tag{5}$$

$$u(x) = x^{\frac{1}{2}}, \tag{6}$$

$$u(x) = x^{-\frac{1}{2}}, \tag{7}$$

$$u(x) = x^{-\frac{1}{4}}, \tag{8}$$

The statement "possible" refers to the possibility that the norm as evaluated may not be finite.