MTH 655, Winter 2017, Assignment 2

In this HW, in the theoretical part, you practice working with Sobolev spaces. In the practical part, you learn to work with the typical structure of the finite element code. We work with the 2-pt BVP

$$-u'' = f, \quad x \in (0,1) \tag{1a}$$

$$u(0) = 0, \quad u(1) = 0 \tag{1b}$$

and with the singular problem from HW 1.

Below the symbol $H(x)$ denotes the Heaviside function

$$H(x) = 1/2sgn(x) + 1/2 = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

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1. Sobolev spaces If (A) is too easy, do (B) instead. Or do both for extra practice.

(A) Determine ∂u, and $\partial^2 u$, on $(-1,1)$, and calculate $\| u \|_{W^{1,1}}$ for

$$u(x) = \begin{cases} x, & x \geq 0 \\ x^2, & x < 0 \end{cases} = u(x) = xH(x) + x^2 H(-x), \tag{2}$$

(B) Find α, β, γ, for

$$u(x) = \gamma x H(x) + H(-x)(\alpha x^2 + \beta), \tag{3}$$

so that u is in $C^0(-1,1), C^1(-1,1), C^2(-1,1), L^2(-1,1) = H^0(-1,1), H^1(-1,1), H^2(-1,1)$. (Consider each space separately but keep your answer concise.)

Hint: remember $C^2 \subsetneq C^1 \subsetneq C^0$, $H^2 \subsetneq H^1 \subsetneq H^0$ which follow from the definitions of the spaces. In addition, in 1d, Sobolev embedding theorems guarantee $H^k(-1,1) \subsetneq C^{k-1}(-1,1)$, for $\mathbf{N} \ni k \geq 1$.

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2. Finite element solution. Grid with three elements as in Handout 3

Get acquainted with `fem1d_2017.m`. Running it with `fem1d_2017(0,1,3,1);` that is, on interval $(0,1)$, with $M = 3$ elements, and uniform degree of piecewise polynomials $p = 1$, produces the errors $\| u - u_h \|_{L^2} \approx 0.062$, and $\| u - u_h \|_{H^1} \approx 0.66$.

(A) Convince yourself that the method converges, with first order in H^1 norm, and second order in L^2. (Produce a table with errors and order α and/or log-log plot of errors when varying $h = \frac{1-0}{M}$).

(B) Change the order of polynomials to $p = 2$ and repeat (A). What is the order of convergence. (You must provide the missing parts of the code in the function `shape`).

C) Change the problem you are solving so that the true solution is $u(x) = x(1-x) + \pi$ (you must recalculate f and modify the code in appropriate places.) What is the order of convergence for piecewise linears and piecewise quadratics?

Extra credit: implement, test, and report on the order of convergence in other norms such as $L^{\inf}, W^{1,1}$.

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3. Use non-uniform grid and polynomial degree
If you run the code with

```
fem1d_2017(0,1,[0,0.3,0.9],[2 1 1]');
```

you will have used nonuniform grid as in Handout 3, with $E_1 = [0, 0.3], E_2 = [0.3, 0.9]$, $E_3 = [0.9, 1]$, and order of polynomials $P|_{E_1} = \mathbf{P}_2$, $P|_{E_2} = \mathbf{P}_1$, $P|_{E_3} = \mathbf{P}_1$. This may not be the best choice, as you see from the solution.
Improve it! Assume you are only allowed to have 3 interior degrees of freedom (total of 5 if you include boundary points). Choose a grid and polynomial degrees so that the error $\| u - u_h \|_{H^1} \leq 0.1$ (there are many possible solutions to this problem).

Extra credit: Guide your quest for the optimal grid and calculate, for each of the element endpoints in the interior, the following "error indicator", defined as the jump of the normal derivative from element to element $[\frac{\partial u_h}{\partial \eta}]$ which in 1d equals $[u'_h]$. This indicator helps to indicate where the local value of $h \| u \|_{H^2}$ is large, and thus where h should be decreased. (*We will later discuss the theory of a–posteriori error estimates which explains why this works*).

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4. Solve the singularly perturbed problem from HW 1, with FEM
To handle the problem

$$-\varepsilon u'' + u = 1, \quad x \in (0,1) \tag{4a}$$

$$u(0) = 0, \quad u(1) = 0 \tag{4b}$$

To solve this problem, you must be able to include in the code the mass matrix with entries $B_{i,j} = \int_0^1 \psi_i \psi_j$. Implement this. (**Hint:** Do not make the problem too hard: you need only to modify the code in

```
m = m + aval * (dpsi)*(dpsi')/dx/dx * w(l)*dx;
```

Show me how you modified the code, and that your solution is correct and converging as $h \downarrow 0$. (Compare with HW 1 and with the FD solution). Be concise.

Extra credit: Discuss all the fine points of the difference between the FD solution and FEM solution. In particular, note that comparing the $\| u - u_h^{FD} \|_{l_\infty}$ to $\| u - u_h^{FEM} \|_{H^1}$ is a bit like comparing apples to oranges. Propose how to set up a fair comparison, and implement it.