MTH 655, Winter 2017, Assignment 5, theory part

Solve two out of the following three.

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1. Element calculations

In the ACF journal paper associated with `fem2d.m` and `femd2d_heat.m`, you can find the code for stiffness matrix calculations in `stima3.m` and `stima4.m`. There is also code for mass matrix calculations for triangles. But the code for some of the local calculations may not be there.

Solve A, or B, or C, or D, or more than one for extra credit.

In all subproblems compare with the use of numerical integration, adapting the 1d Gaussian quadrature to this 2d tensor-product domain case.

(A) Calculate the local stiffness matrix $(a_{\hat{K}}(\psi_i, \psi_j))_{i,j=1}^4 = \int_{\hat{K}} \nabla\psi_j \cdot \nabla\psi_j$ for the bilinear element on the reference square $\hat{K}$. **Hint:** take advantage of a lot of symmetry present in this problem, and of the tensor product definition of the basis functions, which makes the integration really easy.

Extra: Compare with the calculations provided in `stima4.m`

(B) Calculate the local mass matrix $(c_{\hat{K}}(\psi_i, \psi_j))_{i,j=1}^4 = \int_{\hat{K}} \psi_j \psi_j$ for the bilinear element on the reference square $\hat{K}$.

(C) Given a vector $B(x, y)$, assume $\nabla \cdot B = 0$. For the case $B = (1, 1)$, calculate the local matrix $(b_{\hat{K}}(\psi_i, \psi_j))_{i,j=1}^4 = \int_{\hat{K}} \nabla \cdot (B\psi_j)\psi_j$ for the bilinear element on the reference square $\hat{K}$.

(D) Compare the calculations done in class and ACF paper to the code provided in `stima3.m`.

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2. Existence/uniqueness, and variational form

Explore the examples below by solving the differential equation with the given boundary conditions. Does the solution exist. If it does, is it unique ? Also, derive the variational form and check the bilinear form for coercivity on the space that you choose.

Consider $-u'' + c^2 u = f$ on $(0, 1)$, with the constant $c \geq 0$. and the (constant) rhs function $f = 0$, or $f = 1$ (Your answer may depend on which function you are using). Boundary conditions are as in the examples below.

(A) (Homogeneous Neumann problem) Use $c = 0$, $u'(0) = u'(1) = 0$.

(B) (Homogeneous Neumann problem) Use $c \neq 0$, $u'(0) = u'(1) = 0$.

(C) (Periodic problem) Use $c = 0$, $u(0) = u(1), u'(0) = u'(1)$.

3. Projections

Here we explore the notion of L^2 projections and of Ritz (H_0^1) projections. These are an important theoretical tool in the error analysis for parabolic problems.

Let $V = H_0^1(\Omega)$, for an open bounded domain Ω with a sufficiently smooth boundary. Let us be given FE space $V_h \subseteq V$ which uses piecewise Lagrange polynomials of degree k.

For a given $w \in V$, we define the L^2 projection $\Pi_h w$ as that which satisfies

$$(w - \Pi_h w, v_h) = 0, \quad \forall v_h \in V_h. \tag{1}$$

We also define the Ritz (elliptic) projection $R_h w$ by

$$a(w - R_h w, v_h) = 0, \quad \forall v_h \in V_h. \tag{2}$$

where $a(w, v) = \int_\Omega \nabla w \cdot \nabla v$.

(A) **Describe** how you would find $\Pi_h w$ and $R_h w$. **Hint:** start by writing $\Pi_h w = \sum_j w_j \psi_j$. Substitute in (1) to derive the linear system for w_j. Same for (2).

(B) **Prove** the error bounds for $\| w - \Pi_h w \|_{L^2}$ and $\| w - R_h w \|_V$. **Hint:** Follow the same route as when proving error estimates for FE solution. (error expansion, G-O, & interpolation estimates).

(C) **Extra:** what can you say about the error, if $w \in L^2 \setminus H_0^1$ or $w \in H^1 \setminus H_0^1$?