MTH 655, Winter 2017, Assignment 6

Pick one thrust or more. For the chosen thrust, solve at least one of (A) and one of (B).

These are open ended assignments. Spread your wings if you can.

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1. Adaptivity for a singular problem

Consider an L shaped domain Ω with a concave corner of angle $\omega = 3\pi/2$, and a function $u(r,\theta) = r^\gamma sin(\gamma\theta)$, for $0 \le r < 1$ and $0 < \theta < \omega$. This is a classical example for adaptivity.

The theory [CJ, pbm 4.7,] and [CJ, Ex. 4.5] says that since the solution to this problem is not regular ($u \notin H^2$), the error estimates are not optimal. In particular, we only have

$$\| u - u_h \|_{L^2} = O(h^{4/3-\epsilon}) \tag{1}$$

$$\| u - u_h \|_{H^1} = O(h^{2/3-\epsilon}) \tag{2}$$

(A1) Show that u is harmonic. What boundary conditions does it satisfy?
or/and
(A2) Prove (pbm 4.7) (1)–(2)
(B1) Set-up experiments to demonstrate (1)–(2) numerically. Use uniform grid.
or/and
(B2) Set-up an adaptive gridding strategy that, for a fixed numer of degrees of freedom, would give smaller error than uniform grid.

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2. Time-dependent problem

In the ACF journal paper associated with `femd2d_heat.m`, you can find the explanations and the code for the Backward-Euler scheme for the heat equation on the domain Ω

$$u_t - \triangle u = 0 \tag{3}$$

complemented by appropriate boundary conditions, and by the initial condition $u(x,0) = u_{init}(x)$.

Consider also the wave equation

$$u_{tt} - \triangle u = 0 \tag{4}$$

with the same boundary and initial condition as above, plus $u_t(x,0) = 0$.

(A) Assume you know the eigenfunction $u_n(x)$ corresponding to the eigenvalue λ_n for the operator $-\triangle$, i.e., $-\triangle u_n = \lambda_n u_n$, with the same boundary conditions as those applied to the heat equation (3). Show that the function $u^H(A,B;x,t) = cAexp(-\lambda_1 t)u_1(x) + Bexp(-\lambda_5 t)u_5(x)$ satisfies (3), for any A, B. What initial condition does it satisfy?

Propose the solution $u^W(x,t)$ to the wave equation (4).

(B1) Find the eigenvalues λ_1, λ_5 and the eigenfunctions $u_1(x)$ and $u_5(x)$ for the region Ω you worked on in Lab 3. (*The process is the same as in Assignment 4, pbm 2, except that now you are interested both in the eigenvalues, and in eigenfunctions, and now you do not know the true values!*)

and/or

(B2) Solve (3) numerically so that the true solution is $u^H(1, 0.2; x, t)$. Determine the order of convergence as discussed in class.

and/or

(B3) Implement Crank-Nicholson scheme and repeat (B2). Compare to the BE scheme.

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3. Stokes vs Darcy

(A) Play with the BNB conditioms. Consider $A = [1, -1; -1, 1]$, and $B = [1, 1]$. Convince yourse lf that (Au, u) is not coercive on $\mathbb{R}^2$. Is it coercive on $KerB$? Is the saddle point problem with A and B uniquely solvable? Extra: does the inf-sup condition hold for (p, Bv) ?

(B) Solve a Stokes flow problem, and Darcy flow problem on a domain of your choice, preferably a **really** complex one, with boundary conditions of your choice, that would reasonably compare between darcy and Stokes. Compare the velocities obtained for both. Document which FE you are using, and which software you are using.

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4. Challenge: your important problem

Pick a complex problem important in your research, and which is at least as complex as any of the problems above.

Use FE to solve it. (A) Discuss the error and error estimation. Discuss what could go wrong and how you would resolve it. (B) Discuss the implementation details.

Enjoy.