Final Exercise

We showed that $V = \{u \in L^2(a,b) : \ \partial u \in L^2(a,b) \text{ and } u(a) = 0\}$ is a Hilbert space with the scalar product

$$(u,v)_V = \int_a^b (uv + \partial u \partial v)\, dx.$$

Exercise 1. Show that for each $f \in L^2(a,b)$ there is a unique

$$u \in V : \ \int_a^b (uv + \partial u \partial v)\, dx = \int_a^b fv\, dx \ \forall v \in V. \tag{1}$$

Exercise 2. Denote (1) by $u = G(f)$. Show this is equivalent to a boundary-value problem.

Exercise 3. Show that $G \in \mathcal{L}\left(L^2(a,b), V\right)$ and that G is one-to-one.

Exercise 4. Show that $G \in \mathcal{L}(L^2(a,b))$ is self-adjoint and compact. Hint: Let $u = G(f)$, $v = G(g)$, compute $(f, G(g))_{L^2} = (f,v)_{L^2}$ and $(g, G(f))_{L^2} = (g,u)_{L^2}$. Recall that the identity $H^1(a,b) \to L^2(a,b)$ is compact.

Exercise 5. Compute the eigenvalues and eigenfunctions for G. Hint: Note $G(u) = \mu u$ is equivalent to $G(\lambda u) = u$ with $\lambda = \mu^{-1}$, and use Exercise 2.

Exercise 6. Find the range $Rg(G)$ and show it is a Hilbert space with the scalar-product

$$(u,v)_{H^2} = \int_a^b (uv + \partial u \partial v + \partial^2 u \partial^2 v)\, dx.$$

Show that $G \in \mathcal{L}\left(L^2(a,b), H^2(a,b)\right)$ Hint: Show that the graph of G is closed.