PDE FINAL EXERCISE

May, 2015

Let G be a smoothly bounded domain in $\mathbb{R}^n$ for which there is a well defined boundary trace operator $\gamma: \ H^1(G) \to L^2(\partial G)$. Let the boundary be partitioned into disjoint connected pieces, $\partial G = \Gamma_D \cup \Gamma_R$. Let $F(\cdot) \in L^2(G)$ be given along with the functions $a = a(x)$ and $\mathbf{b} = \mathbf{b}(x)$.

1. State a weak formulation of the boundary-value problem

$$-\nabla \cdot (a(x)\nabla p - \mathbf{b}(x)\, p) = F(x) \text{ in } G\,, \tag{1a}$$

$$p = 0 \text{ on } \Gamma_D\,, \quad a\frac{\partial p}{\partial n} - \mathbf{b}\cdot\mathbf{n}p = 0 \text{ on } \Gamma_R\,, \tag{1b}$$

and show it has a unique solution. Specify the function spaces used, and the conditions assumed on the functions $a(x)$ and $\mathbf{b}(x)$. Reference the results that you use from HSM.

2. State a weak formulation of the boundary-value problem

$$-\nabla \cdot (a(x)\nabla u - \mathbf{b}(x)u) = F(x) \text{ in } G, \tag{2a}$$

$$u = 0 \text{ on } \Gamma_D, \quad a\frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_R, \tag{2b}$$

and show it has a unique solution. Specify the function spaces used, and the conditions assumed on the functions $a(x)$ and $\mathbf{b}(x)$. Reference the results that you use from HSM.

3. Show the initial-boundary-value problem

$$\tfrac{\partial u}{\partial t} - \nabla \cdot (a(x)\nabla u - \mathbf{b}(x)u) = F(x,t) \text{ in } G, \tag{3a}$$

$$\text{either (1b) or (2b)}, \tag{3b}$$

$$u(x,0) = u_0(x) \text{ in } G, \tag{3c}$$

has a unique solution and state the conditions assumed on the functions $a = a(x)$, $\mathbf{b} = \mathbf{b}(x)$, $u_0(x)$, and $F(x,t)$.