Exercise # 1 Due October 26.

Let $A_j(\cdot,\cdot) : G \times \mathbb{R}^{N+1} \to \mathbb{R}$ for each $0 \le j \le N$ and assume $A_j(x,\xi) \in \mathbb{R}$ is continuous in $\xi \in \mathbb{R}^{N+1}$ and measurable in $x \in G$, $1 < p < \infty$, $k(\cdot) \in L^{p'}(G)$, and

$$|A_j(x,\xi)| \le C \sum_{i=0}^{N} |\xi_i|^{p-1} + k(x),$$
$$\sum_{j=0}^{N} A_j(x,\xi)\xi_j \ge c_0 \sum_{j=0}^{N} |\xi_j|^p - k(x),$$
$$\sum_{j=0}^{N} (A_j(x,\xi) - A_j(x,\eta))(\xi_j - \eta_j) \ge 0.$$

For each $\mathbf{u} \in L^p(G, \mathbb{R}^{N+1})$, that is, $\mathbf{u}(x) = (u_0(x), u_1(x), ..., u_N(x))$ with each $u_i \in L^p(G)$, denote the composite function $x \mapsto A_j(x, \mathbf{u}(x))$ by $A_j(\mathbf{u})$.

- Show that $A_j(\mathbf{u}) \in L^{p'}(G)$ for each $\mathbf{u} \in L^p(G)$.
- Show that $\mathbf{u} \in L^p(G, \mathbb{R}^{N+1}) \mapsto A_j(\mathbf{u}) \in L^{p'}(G)$ is continuous. (Hint: Use the Nemytskii Theorem.)
- Show that $u \in W^{1,p}(G) \mapsto A_j(u, \boldsymbol{\nabla} u) \in L^{p'}(G)$ is continuous.

Define $\mathcal{A} : W^{1,p}(G) \to W^{1,p}(G)'$ by

$$\mathcal{A}(u)(v) = \int_G \sum_{j=0}^{N} A_j(x, u(x), \boldsymbol{\nabla} u(x)) \partial_j v(x)\, dx, \quad u,\ v \in W^{1,p}(G).$$

where $\partial_0 = Id$.

- Show that $\mathcal{A}$ is continuous, bounded, coercive, and monotone.