Exercise Set # 3 Due December 4.

Assume B is m-accretive in the Hilbert space H and consider the initial-value problem

$$\tfrac{d}{dt}u(t) + B(u(t) \ni 0, \quad u(0) = u_0. \tag{1}$$

Let $\mathcal{B}$ denote the realization of B in the space $\mathcal{H} = L^2((0,1), H)$ (see Example 2.C, p. 164), and let $\mathcal{A} = \frac{d}{dt}$ with $u(0) = u_0$ in the space $\mathcal{H} = L^2((0,1), H)$ (see Example 2.D, p. 164). Then (1) is given by $\mathcal{A}(u) + \mathcal{B}(u) \ni 0$ in $\mathcal{H}$.

In order to approximate the solution of (1), let $\alpha > 0$ and solve $\mathcal{A}(u) + \mathcal{B}_\alpha(u) \ni 0$, that is,

$$\tfrac{d}{dt}u(t) + B_\alpha(u(t)) \ni 0, \quad u(0) = u_0. \tag{2}$$

Define a second variable by $v(t) = (I + \alpha B)^{-1}u(t)$ to obtain the system

$$\tfrac{d}{dt}u(t) + \tfrac{1}{\alpha}(u(t) - v(t)) = 0, \quad u(0) = u_0, \tag{3a}$$
$$B(v(t)) + \tfrac{1}{\alpha}(v(t) - u(t)) \ni 0, \tag{3b}$$

which is equivalent to (2). By eliminating u from (3), we obtain the initial-value problem

$$\tfrac{d}{dt}(v + \alpha B(v)) + B(v) \ni 0, \quad (v + \alpha B(v))|_{t=0} = u_0. \tag{4}$$

Exercise 1.

- Show that (2) has a unique solution, and determine for what class of initial values u_0 this holds.
- Show that (2) is equivalent to (3)
- Show that (3) is equivalent to (4).

Exercise 2. A second approximation of (1) is obtained similarly by

$$\mathcal{A}_\alpha(u) + \mathcal{B}(u) \ni 0, \tag{5}$$

- Introduce a second variable $w = (I + \alpha\mathcal{A})^{-1}u$ to obtain a system similar to (3).
- Show that (5) has a unique solution.

Exercise 3. Let G be a smoothly bounded domain in $\mathbb{R}^N$, $\varphi : \mathbb{R} \to \mathbb{R}$ a proper convex lower semicontinuous function, and on the space $H = L^2(G)$ define the proper convex lower semicontinuous function

$$\Psi(u) = \tfrac{1}{2} \int_G |\boldsymbol{\nabla} u(x)|^2 dx + \int_G \varphi(u(x)) dx \,, \tag{6}$$

with domain $dom(\Psi) = \{u \in H_0^1(G) : \varphi \circ u \in L^1(G)\}$.

- Characterize $f \in \partial\Psi(u)$ by a boundary-value problem. (See Example 2.F on p. 167 for comparison.)
- Find the initial-boundary-value problem that characterizes the system (3) with $B = \partial\Psi$