The complex-valued exponential

How should we define the complex-valued function e^{it} for real values of t so that it behaves like an exponential should? That is, we want the complex function $u(t) = e^{it}$ to satisfy the *initial-value problem*

$$u'(t) = iu(t), \text{ and } u(0) = 1. \tag{1}$$

The complex function $u(t)$ can be written as $u(t) = x(t) + iy(t)$ for a unique pair of real-valued functions, and our objective is to describe these two functions. Substituting $x(t) + iy(t)$ for $u(t)$ in (1) and comparing the real and imaginary parts show that $x(t)$ and $y(t)$ must satisfy the system

$$\begin{aligned} \dot{x}(t) &= -y(t), \quad x(0) = 1, \\ \dot{y}(t) &= x(t), \quad y(0) = 0. \end{aligned} \tag{2}$$

Note that $\frac{d}{dt}(x^2(t) + y^2(t)) = 0$, so $x^2(t) + y^2(t)$ is a constant. The initial conditions in (2) show the constant is one, so we have

$$x^2(t) + y^2(t) = 1. \tag{3}$$

In fact, the pair $(x(t), y(t))$ traces out the unit circle in the plane with speed equal to one: $\sqrt{(\dot{x}(t))^2 + (\dot{y}(t))^2} = 1$, and we check the signs of derivatives to determine the direction is counterclockwise starting at $(1, 0)$.

In the right half-plane, we determine from (2) and (3) that

$$\frac{dy}{dt} = \sqrt{1 - y^2}, \quad y(0) = 0,$$

and we separate and integrate this first order equation to obtain

$$\int_0^y \frac{ds}{\sqrt{1 - s^2}} = t, \quad -1 \leq y \leq 1. \tag{4}$$

The identity (4) defines the inverse function, $t = \arcsin(y)$ on the unit interval. Thus, $y(t) = \sin(t)$ and from (2) we get $x(t) = \cos(t)$. The same holds in the left half-plane, so the complex exponential is

$$e^{it} = \cos(t) + i \sin(t).$$