1. Linear Algebra Basics

We begin with a discussion of the solvability of a system of 2 equations in 2 unknowns:

$$
\begin{aligned}
ax + by &= f,\\
cx + dy &= g.
\end{aligned} \tag{1}
$$

The corresponding *homogeneous* system is

$$
\begin{aligned}
ax + by &= 0,\\
cx + dy &= 0,
\end{aligned} \tag{2}
$$

and it plays an important role. The *coefficients* of the system are given as the 2×2 matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We shall always assume that the matrix is not all zeros, that is, at least one of the four constants is non-zero. The matrix M is called *singular* if $ad - bc = 0$.

Assume M is singular. Then (2) has a solution given by either $[x, y] = [-b, a]$ or $[x, y] = [-d, c]$, so we conclude that there is a non-zero solution of the homogeneous system (2).

Assume M is singular. If there is a solution of (1), then we necessarily have $-cf + ag = 0$ and $-df + bg = 0$, so we conclude that the non-homogeneous system (1) does *not* have a solution for *every* pair $[f, g]$.

Suppose that M is non-singular. Then we can always solve (1) for the unique solution pair $[x, y]$: first eliminate y and then eliminate x to obtain explicit formulas for the two components. Thus, we see that for every pair $[f, g]$, the system (1) has exactly one solution. In particular, if $[f, g] = [0, 0]$, then the only solution is $[x, y] = [0, 0]$.

These results are summarized in the following.

Theorem 1. *The matrix M is non-singular*
if and only if
for every pair $[f, g]$ the system (1) *has exactly one solution,*
and this holds if and only if
the only *solution of the homogeneous system* (2) *is the pair $[x, y] = [0, 0]$.*

Also, the matrix M is singular if and only if
the homogeneous system (2) *has a non-zero solution. That is, there is a*

pair $[c_1, c_2]$, *not both zero, such that*

$$c_1 \begin{bmatrix} a \\ c \end{bmatrix} + c_2 \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{3}$$

Definition 1. *If* (3) *holds for a pair* $[c_1, c_2]$, *not both zero, we say that the vectors* (a, c), (b, d) *are* linearly dependent.

That is, the pair (a, c), (b, d) is *linearly independent* if the equation (3) implies that $c_1 = 0$ and $c_2 = 0$.

2. The Initial-Value Problem

Our objective is to describe the set of solutions of the initial-value problem

$$y''(t) + p(t)y'(t) + q(t)y(t) = f(t), \quad a < t < b, \tag{4a}$$

$$y(t_0) = y_0, \quad y'(t_0) = y_1. \tag{4b}$$

The equation (4a) is a general second-order *linear differential equation*, and the two equations (4b) are the *initial conditions*. The discussion depends on the following *existence-uniqueness theorem*:

Theorem 2. *Assume that the functions* $p(t)$, $q(t)$, $f(t)$ *are continuous on the interval* $a < t < b$ *and that* t_0 *is a point in this interval. Then for any pair of numbers,* y_0, y_1, *there is exactly one solution of the initial-value problem* (4), *and the solution exists on the entire interval.*

Note that the set of functions on the interval (a, b) is a linear space, so the notion of *linearly independent* applies as well to them. Also, for any function $y(t)$ on the interval (a, b), we shall use the expression

$$L[y] = y''(t) + p(t)y'(t) + q(t)y(t)$$

to abbreviate computations. Note that this is a *linear operator*, in the sense that for any pair of functions, $y_1(t), y_2(t)$ and constants c_1, c_2, we see from a direct computation that

$$L[c_1 y_1 + c_2 y_2] = c_1 L[y_1] + c_2 L[y_2]. \tag{5}$$

2.1. **Homogeneous equation.** We begin by describing the general solution of the *homogeneous equation*

$$y''(t) + p(t)y'(t) + q(t)y(t) = 0\,. \tag{6}$$

Note that the identity (5) implies that if $y_1(t)$ and $y_2(t)$ are solutions to (6), then for any pair of constants, $c_1,\ c_2$, the linear combination $c_1y_1(t) + c_2y_2(t)$ is also a solution of (6).

Definition 2. *Let the functions $y_1(t)$ and $y_2(t)$ be solutions of* (6)*. The* Wronskian *of these solutions is the function*

$$W(y_1, y_2; t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)y_2'(t) - y_2(t)y_1'(t). \tag{7}$$

Note that for any t we have $W(y_1, y_2; t) = 0$ if and only if the matrix $\begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix}$ is singular. We shall apply the linear algebra results above.

Suppose that there is a t_0 in (a, b) for which $W(y_1, y_2; t_0) = 0$. Then there is a pair of constants $[c_1, c_2]$, not both zero, such that

$$\begin{aligned} c_1y_1(t_0) + c_2y_2(t_0) &= 0, \\ c_1y_1'(t_0) + c_2y_2'(t_0) &= 0. \end{aligned} \tag{8}$$

This means that the function $y(t) = c_1y_1(t) + c_2y_2(t)$ is a solution of the initial-value problem (4) with $f(t) = 0,\ y(t_0) = 0,\ y'(t_0) = 0$. But the zero function is a solution of this same initial-value problem, so by the uniqueness part of Theorem 2, we must have $y(t) = 0$. That is, we have shown there is a pair of constants, $[c_1, c_2]$, not both zero, such that

$$c_1y_1(t) + c_2y_2(t) = 0, \quad a < t < b.$$

This says that the pair of functions $y_1(t)$ and $y_2(t)$ is linearly dependent. On the other hand, if the functions $y_1(t)$ and $y_2(t)$ are linearly dependent, then we can differentiate and obtain the second of the two equations

$$\begin{aligned} c_1y_1(t) + c_2y_2(t) &= 0, \\ c_1y_1'(t) + c_2y_2'(t) &= 0, \end{aligned} \tag{9}$$

for a pair of constants, $[c_1, c_2]$, not both zero, and this implies that $W(y_1, y_2; t) = 0$ for *all* t in the interval (a, b). Thus we have shown that $W(y_1, y_2; t_0) = 0$ for *some* t_0 in the interval (a, b) if and only if $W(y_1, y_2; t) = 0$ for *all* t in the interval (a, b), and this holds if and only if the pair of solutions $y_1(t)$ and $y_2(t)$ is linearly dependent.

Now suppose that t_0 is a point in (a, b) for which $W(y_1, y_2; t_0) \neq 0$, and let $y(t)$ be *any* solution of (6). Since the matrix of coefficients is non-singular, we can solve the system

$$\begin{aligned} c_1 y_1(t_0) + c_2 y_2(t_0) &= y(t_0), \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) &= y'(t_0). \end{aligned} \tag{10}$$

for a unique pair of constants $[c_1, c_2]$. The system (10) implies that the two functions $c_1 y_1(t) + c_2 y_2(t)$ and $y(t)$ are both solutions of the same initial-value problem (4) with $f(t) = 0$. But then we must have (by uniqueness of solutions)

$$y(t) = c_1 y_1(t) + c_2 y_2(t), \ a < t < b. \tag{11}$$

Thus, *every* solution of the homogeneous equation (6) is given by (11) for a corresponding pair of constants, $[c_1, c_2]$.

Definition 3. *If all solutions of the homogeneous equation* (6) *are given by linear combinations of the two solutions* $y_1(t)$ *and* $y_2(t)$*, we say that these two solutions are a* fundamental set *or* basis *of solutions of* (6)*.*

We summarize this in the following.

Theorem 3. *Let the functions* $y_1(t)$ *and* $y_2(t)$ *be solutions of* (6)*. The Wronskian* $W(y_1, y_2; t)$ *is either identically zero or never zero on the interval* (a, b)*.*

In the first case, i.e.*, if there is a* t_0 *in* (a, b) *for which* $W(y_1, y_2; t_0) = 0$*, then the two solutions* $y_1(t)$ *and* $y_2(t)$ *are linearly dependent.*

In the second case, the pair $y_1(t)$ *and* $y_2(t)$ *is a fundamental set of solutions of* (6)*. In particular, for any choice of initial values,* y_0*,* y_1*, there is a unique pair of constants,* $[c_1, c_2]$ *for which* (11) *is the solution of the initial-value problem consisting of the homogeneous equation* (6) *together with the initial conditions* (4b)*.*

An algorithm for representing all solutions to (6) follows immediately. Find a fundamental set of solutions to (6), and then the *general solution* is represented as linear combinations of these two solutions. The Wronskian provides a means to test these to determine whether they are a fundamental set or they are linearly dependent.

2.2. **Non-homogeneous equation.** Now we can describe the solutions of the non-homogeneous equation (4a). Suppose that we have a solution $y_p(t)$ of the non-homogeneous equation (4a). If $y(t)$ is any solution of (4a), then the difference satisfies $L[y(t) - y_p(t)] = L[y(t)] - L[y_p(t)] = f(t) - f(t) = 0$, so $y(t) - y_p(t)$ is a solution of the homogeneous equation (6). As such, it must be a linear combination of the fundamental set $y_1(t),\ y_2(t)$, so we have

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + y_p(t),\ a < t < b. \tag{12}$$

Theorem 4. *Let the functions $y_1(t)$ and $y_2(t)$ be a fundamental set of solutions of the homogeneous equation* (6)*, and let $y_p(t)$ be a solution of the non-homogeneous equation* (4a)*. Then the general solution of* (4a) *is given by* (12) *for an appropriate pair of constants, and for any choice of initial values, $y_0,\ y_1$, there is a unique pair of constants, $[c_1, c_2]$ for which* (12) *is the solution of the initial-value problem* (4a)*.*

Thus, in order to find all solutions of (4a), or to find the solution of the initial-value problem (4), it suffices to find a fundamental set of solutions to the homogeneous equation (6) and one solution to the non-homogeneous equation (4a).