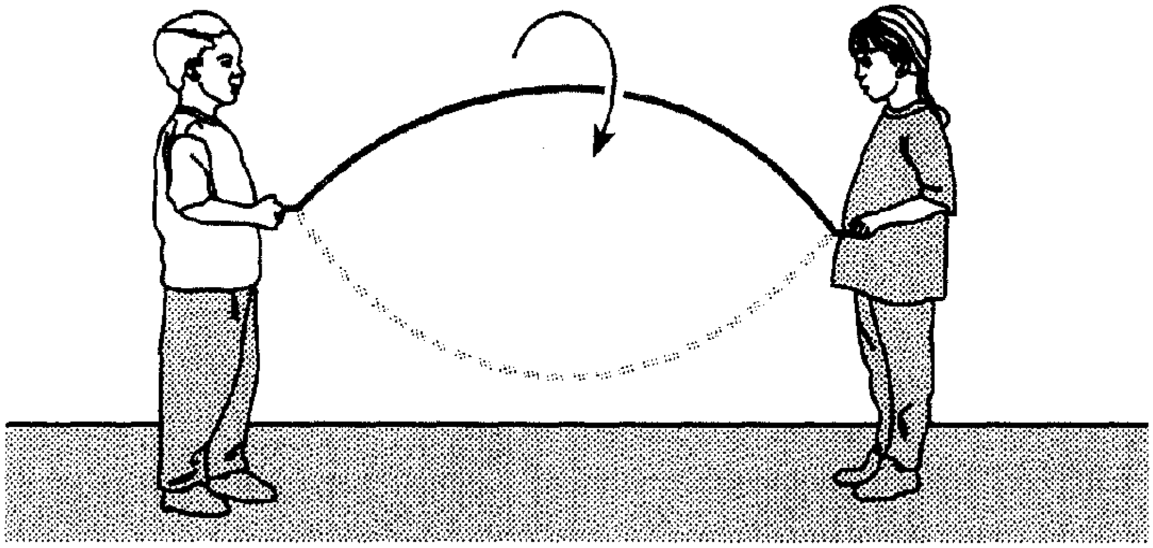
3.9 Rotating string *(page 162)*

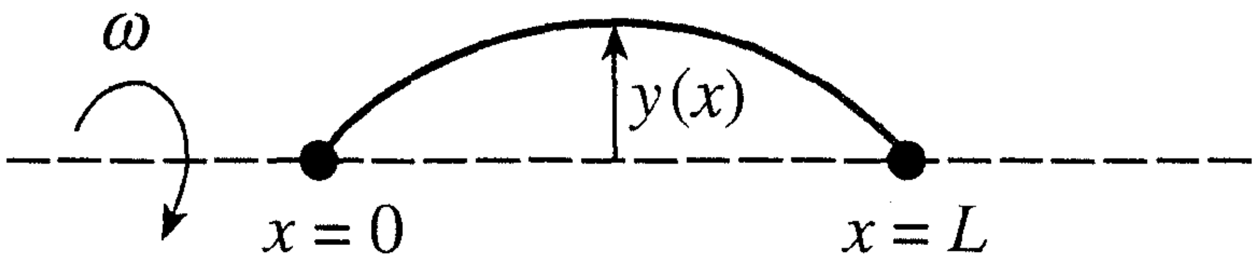
Problem: For which value(s) of the angular velocity ω will the string (rope) be deflected?

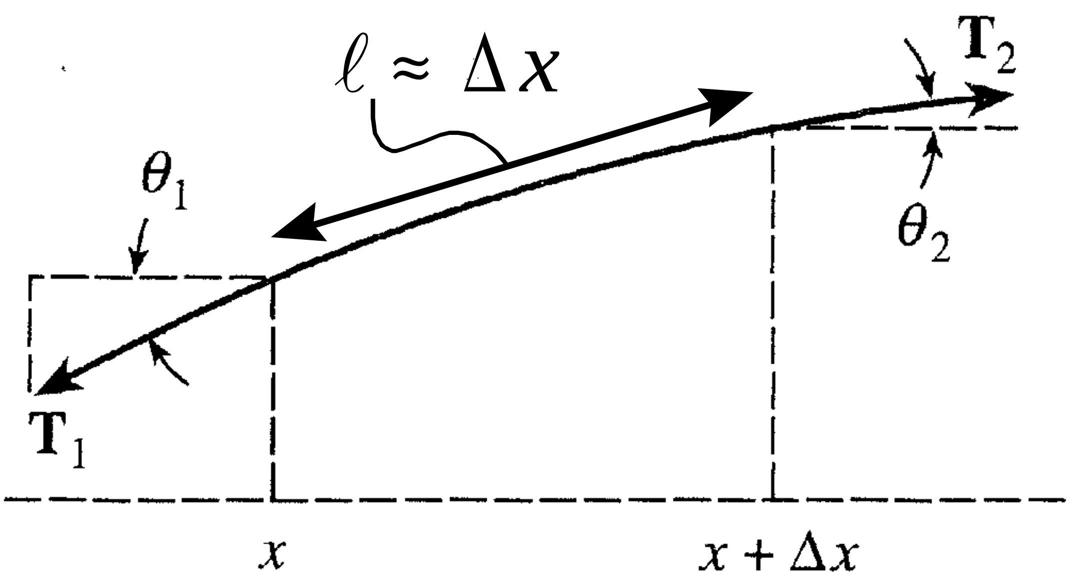


Assumptions:

(1) The angular velocity ω is constant
(2) The tension T in the rope is constant and large – gravitation can therefore be ignored
(3) The rope is uniform – the mass per unit length ρ is therefore constant
(4) The deflections are small
(5) The endpoints are fixed

Rotating string





Newton's second law: $(\downarrow)\,\Sigma F_y = ma$

$$T_1 \sin\theta_1 - T_2 \sin\theta_2 = \overbrace{(\rho\,\Delta x)}^{m} \times \overbrace{(y\omega^2)}^{a=a_n}$$

$$\text{but} \quad T_1 = T_2 = T$$

$$\rho\,\Delta x\, y\,\omega^2 = T(\sin\theta_1 - \sin\theta_2)$$

but $\sin\theta_1 \approx \tan\theta_1$ and $\sin\theta_2 \approx \tan\theta_2$,

since θ_1 and θ_2 are small (deflections small)

thus $\sin\theta_1 \approx y'(x)$ and $\sin\theta_2 \approx y'(x+\Delta x)$

$$\rho\, y\, \omega^2 = \frac{T(y'(x) - y'(x+\Delta x))}{\Delta x}$$

$$\rho\, y\, \omega^2 = -T\left(\frac{y'(x+\Delta x) - y'(x)}{\Delta x}\right)$$

but $\displaystyle \lim_{\Delta x \to 0} \frac{y'(x+\Delta x) - y'(x)}{\Delta x} = y''(x)$

thus $\displaystyle \rho\, y\, \omega^2 = -T\frac{d^2y}{dx^2}$

$$\Rightarrow \quad \frac{d^2y}{dx^2} + \frac{\rho\omega^2}{T}y = 0$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} + \lambda y = 0 \quad \text{where} \quad \lambda = \frac{\rho\omega^2}{T}$$

with boundary cond's $y(0) = 0$ and $y(L) = 0$

(As before) Non-trivial solutions when:

$$\lambda_n = \frac{n^2\pi^2}{L^2} = \frac{\rho\,{\omega_n}^2}{T}, \; n = 1, 2, \ldots$$

Thus critical angular velocities: $\omega = \omega_n = \frac{n\pi}{L}\sqrt{\frac{T}{\rho}}$

Non-trivial solutions: $y_n = C\,\sin\left(\frac{n\pi x}{L}\right)$

For $n = 1$ First critical angular velocity

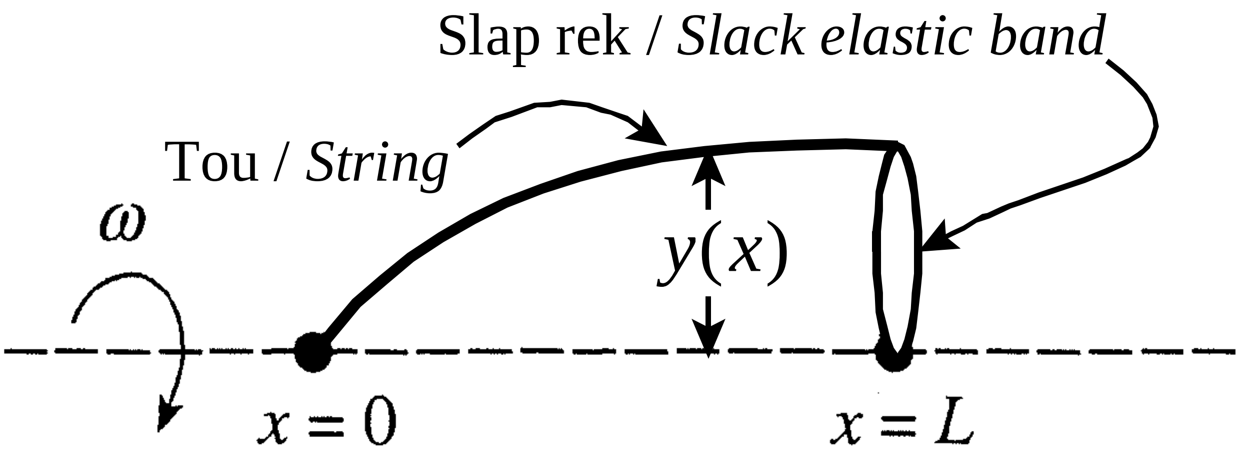
$$\omega_1 = \frac{\pi}{L}\sqrt{\frac{T}{\rho}}$$

First deflection mode: $y_1 = C\,\sin\left(\frac{\pi x}{L}\right)$

$$
\begin{aligned}
0 < \omega < \omega_1 &\Rightarrow \text{ no deflection} \\
\omega = \omega_1 &\Rightarrow \text{ first deflection mode} \\
\omega_1 < \omega < \omega_2 &\Rightarrow \text{ no deflection} \\
\omega = \omega_2 &\Rightarrow \text{ second deflection mode}
\end{aligned}
$$

Show that the unit for ω is rad/s

Rotating string with elastic support



$$\text{DE:} \quad \frac{d^2y}{dx^2} + \lambda y = 0 \quad \text{where} \quad \lambda = \frac{\rho\omega^2}{T}$$

with boundary cond's $y(0) = 0$ and $y'(L) = 0$

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Case I: $\boxed{\lambda = 0}$ Show that $y \equiv 0$ (trivial sol)

Case II: $\boxed{\lambda < 0}$ Show that $y \equiv 0$ (trivial sol)

Case III: $\boxed{\lambda > 0}$ Show that...

Non-trivial solutions only when:

$$\lambda_n = \frac{(2n-1)^2}{L^2}\frac{\pi^2}{4},\ n = 1, 2, 3, \ldots$$

Non-trivial solutions (eigenfunctions):

$$y_n(x) = C\,\sin\left(\frac{\pi(2n-1)}{2L}\,x\right)$$

$\boxed{\text{For } n = 1}$ First critical angular velocity

$$\omega_1 = \frac{\pi}{2L}\sqrt{\frac{T}{\rho}}$$

First deflection mode: $y_1 = C\,\sin\left(\frac{\pi x}{2L}\right)$ (sketch)